Math 520 Spring 2017
Solution to Problem Set 10

1) Suppose that $\lambda_1,\ \lambda_2,\ \lambda_3$ are the three distinct eigenvalues of A so that the eigenspaces corresponding to λ_1, λ_2 respectively are of dimensions $2, 4$ respectively. Since every eigenspace must have dimension at least one, the sum of the dimensions of the eigenspaces is at least $1 + 2 + 4 = 7$ which is already the full dimension of $\mathbb{R}^7$. It follows that there must be a basis for $\mathbb{R}^7$ consisting of eigenvectors of A and thus A is diagonalizable.

2) First note that if $\mu \neq 0$ and $\boldsymbol{v} \neq 0$ as an eigenvector, $\mu\boldsymbol{v} \neq 0$. Moreover,
$A\left(\mu\boldsymbol{v}\right) = \mu(A\boldsymbol{v}) = \mu(\lambda\boldsymbol{v}) = \lambda\left(\mu\boldsymbol{v}\right).$

Therefore, $\mu\boldsymbol{v}$ is an eigenvector of A.

3) For u, v in $\mathbb{R}^n$,

$$\begin{aligned} \|u+v\|^2 + \|u-v\|^2 &= (u+v)\cdot(u+v) + (u-v)\cdot(u-v) \\ &= (u\cdot u + u\cdot v + v\cdot u + v\cdot v) + (u\cdot u - u\cdot v - v\cdot u + v\cdot v) \\ &= 2\,u\cdot u + 2\,v\cdot v = 2\,\|u\|^2 + 2\,\|v\|^2 . \end{aligned}$$

4) First note that by definition,

$$V^\perp = \left\{u \in \mathbb{R}^n:\ u\cdot v = 0\, for\, all\, v \in V\right\}.$$

Clearly, 0 is in $V^\perp$ as its dot product with any vector is zero. For any $u_1,\ u_2 \in V^\perp$ and $c \in \mathbb{R}$, we have
$u_1 \cdot v = u_2 \cdot v = 0$

for all $v \in V$, and it follows that

$$(u_1 + u_2)\cdot v = u_1\cdot v + u_2\cdot v = 0 + 0 = 0\left(cu_1\right)\cdot v = c\,\left(u_1\cdot v\right) = c\,0 = 0.$$

Thus $V^\perp$ is a subspace of $\mathbb{R}^n$.

5) Set $v_2 = \begin{bmatrix}1\\1\\0\end{bmatrix}$, $v_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}$. Then it is easy to see that

$$v_1 \cdot v_2 = v_2 \cdot v_3 = v_1 \cdot v_3 = 0 \ .$$