MATH 520 PRACTICE FINAL
SPRING 2017
BROWN UNIVERSITY
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Name:

Allowed materials are pen, pencil, and straightedge. You have three hours.

Problem 1

Find t such that the solution set of $A\mathbf{x} = \mathbf{0}$ has 8 free variables, where

$$A = \begin{bmatrix} -4 & 3 & 4 & 1 & 4 & 4 & 3 & -3 & -5 & 1 \\ 4 & 4 & 6 & -6 & 1 & 4 & 1 & -3 & -2 & -4t+2 \\ 0 & 7 & 10 & -5 & 5 & 8 & 4 & -6 & -7 & t \end{bmatrix}.$$

Solution

Final answer:

Problem 2

Suppose that A is an $m \times 5$ matrix for which the equation $A\mathbf{x} = \mathbf{0}$ has

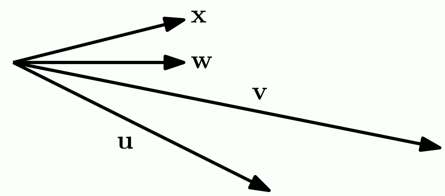
$$\mathbf{x} = \begin{bmatrix} 8 \\ 3 \\ 2 \\ -1 \\ 0 \end{bmatrix}$$

in its solution set. Are the first 4 columns of A linearly independent? Explain your reasoning.

Solution

Problem 3

Two of the following four vectors sum to one of the other two vectors. Write an equation expressing this relationship.



Solution

Final answer:

Problem 4

Find the least number M such that every 3×4 matrix can be reduced to its reduced row echelon form with no more than M row operations. Explain your reasoning.

Solution

Problem 5

Suppose that A is an 5×7 matrix of rank 3, and that B is a 7×4 matrix of rank 2. Show that the rank of AB is no greater than 2.

Solution

Problem 6

Solve the matrix equation $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix}$ for $\begin{bmatrix} x \\ y \end{bmatrix}$ two ways: (i) using the 2×2 matrix inversion formula, and (ii) using Cramer's rule. Show that you get the same answer either way.

Solution

Problem 7

Find c and d so that $A^2 = I$, where $A = \begin{bmatrix} 1 & \frac{1}{3} \\ c & d \end{bmatrix}$.

Solution

Problem 8

Suppose that A is a square matrix. Show that if A^2 is invertible, then A is also invertible.

Solution

Problem 9

Find $\det \begin{bmatrix} 2 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 2 \end{bmatrix}$.

Solution

Problem 10(a)

The set V of all functions from $[0, 1]$ to $\mathbb{R}$, equipped with the usual notions of function addition and scalar multiplication, is a vector space. Consider the subset S of V consisting of those functions which are discontinuous at one or more values of x. Show that S is not a subspace of V.

Solution

Problem 10(b)

Consider the set S' of all elements of V which are discontinuous at at most finitely many points (that is, $f \in S'$ if and only if the discontinuity set of f is empty or finite). Show that S' *is* a subspace of $\mathbb{R}^n$. (Note: recall from calculus that if two functions f and g are both continuous at $x \in [0, 1]$, then $f + g$ is continuous at x).

Solution

Problem 11

Consider the linear transformation $T : \mathbb{P}_{100} \to \mathbb{P}_{100}$ defined by $T(p) = p + p''$. Show that the rank of T is 101. Hint: start by considering the nullity of T.

Solution

Problem 12

Suppose that $A = \begin{bmatrix} 1 & 0 & -1 \\ \sqrt{3} & a & 17 \\ 2 & 0 & b \end{bmatrix}$ has eigenvalues 2 and 3, and that the eigenvalue 2 has multiplicity 2. Find a and b.

Solution

Problem 13

The *spectral theorem*, probably the most important thorem not covered in the course, states that if A is an $n \times n$ matrix with n distinct eigenvalues $\lambda_1, \ldots, \lambda_n$, then

$$A = \lambda_1 P_1 + \cdots + \lambda_n P_n, \tag{1}$$

where P_k is a matrix representing a projection onto the eigenspace corresponding to λ_k, for each $k = 1, 2, \ldots, n$.

To find P_k satisfying (1), we let U be the matrix whose columns are linearly independent eigenvectors of A (in the same order as their corresponding λ's), and we obtain U_k from U by replacing all but the kth column with zeros. Then we let $P_k = U_k U^{-1}$.

Verify the spectral theorem in the case $A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$. (Hint: this problem might seem intimidating, but it's purely a matter of following the instructions and doing some matrix calculations.)

Solution

Problem 14

Use linear algebra to find real numbers α and β which minimize

$$(y_1 - \alpha - \beta x_1)^2 + (y_2 - \alpha - \beta x_2)^2 + (y_3 - \alpha - \beta x_3)^2 + (y_4 - \alpha - \beta x_4)^2,$$

where $\{(x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4)\} = \{(1, 4), (2, 5), (3, 5), (4, 7)\}$. Sketch these ordered pairs in the plane and interpret your findings graphically.

Solution