MATH 520 MIDTERM II
SPRING 2017
BROWN UNIVERSITY
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Name:

This is a pencil-and-paper-only exam. You have two hours.

Problem 1(a) (8 points)

Find $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}^{-1}$

Solution

Final answer:

Problem 1(b) (8 points)

Suppose that A is a 3×3 matrix with the property that $A\mathbf{v}_1 = \mathbf{e}_1$, $A\mathbf{v}_2 = \mathbf{e}_2$, and $A\mathbf{v}_3 = \mathbf{e_2} + \mathbf{e}_3$, where $\mathbf{v}_1 = \begin{bmatrix} -4 \\ 2 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix}$. Find A^{-1}. (Note: $\mathbf{e}_1 \mathbf{e}_2, \mathbf{e}_3$ denote the standard basis vectors in $\mathbb{R}^3$.)

Solution

Final answer:

Problem 2(a) (8 points)

Suppose t is a real number and $A = \begin{bmatrix} t & 13 \\ 2 & t \end{bmatrix}$. Suppose further that the area of the image of any square S in $\mathbb{R}^2$ under the linear transformation $T(\mathbf{x}) = A\mathbf{x}$ is equal to 10 times the area of S. Find all four possible values of t (note: you will receive most of the credit for finding two of them).

Solution

Final answer:

Problem 2(b) (8 points)

Show that $\det(A + B)$ is not always equal to $\det(A) + \det(B)$, where A and B are 2×2 matrices. Hint: just make up some examples for A and B; it will probably work.

Solution

Final answer:

Problem 3(a) (10 points)

The matrices $A = \begin{bmatrix} 1 & 4 & 5 & 8 & 2 \\ 0 & 1 & -2 & 3 & 5 \\ -2 & -7 & -12 & -13 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & 13 & -4 & -18 \\ 0 & 1 & -2 & 3 & 5 \\ 0 & 2 & -4 & 6 & 10 \end{bmatrix}$ are row equivalent.

Find a basis for the row space of A and a basis for the column space of A.

Solution

Final answer:

Problem 3(b) (5 points)

Find the rank and the nullity of A.

Solution

Final answer:

Problem 4(a) (8 points)

Consider the set $S = \{f \in C([0,1]) : f(x) \geq 0 \text{ for all } x \in [0,1]\}$ of continuous functions from $[0,1]$ to $\mathbb{R}$ which are nowhere negative. Show that S **is** closed under vector addition in $C([0,1])$ but is **not** closed under scalar multiplication. So is S a subspace?

Solution

Problem 4(b) (8 points)

Consider the set S of polynomials in $\mathbb{P}_4$ which have no t^2 term. So, for example, $t - 4t^3$ is in the set, while $1 - 8t^2 + t^4$ is not. Is S a subspace of $\mathbb{P}^4$? Justify your answer.

Solution

Problem 5 (10 points)

Consider the linear transformation $T : \mathbb{P}_7 \to \mathbb{P}_7$, where $T(p) = p'$. In other words, T acts by taking the derivative. So, for example, $T(-3t^4 + t^2 - t) = -12t^3 + 2t - 1$.

Find the range and the kernel of T. Feel free to describe these sets using either a verbal description or math notation, as you prefer (as long as they are clearly specified). Explain your reasoning.

Solution

Problem 6(a) (6 points)

Explain why $\det A$ is an integer whenever A is a square matrix with integer entries. Hint: for partial credit, work out $\det A$ for some 3×3 matrix with integer entries. How could you have known the answer would be an integer before you did all the calculations?

Solution

Problem 6(b) (9 points)

Use Cramer's rule to show that if A is an invertible $n \times n$ matrix with integer entries and $\mathbf{b}$ is an $n \times 1$ vector with integer entries, then the unique solution $\mathbf{x}$ of the equation $A\mathbf{x} = \mathbf{b}$ is a vector whose entries are rational numbers (that is, simplified fractions with integer numerator and denominator) **whose denominators evenly divide** $\det A$. (So, for example, if $\det A = 8$, then the denominators of the entries of $\mathbf{x}$ are necessarily in the set $\{1, 2, 4, 8\}$).

Solution

Problem 7 (12 points)

Suppose that W is a ten-dimensional vector space and V is a subspace of W whose dimension is 6. Show that there is a four-dimensional subspace U of W with the property that $U \cap V = \{\mathbf{0}\}$ (in other words, so that U and V have no vectors in common except the zero vector). Explain your reasoning precisely. Hint: begin by considering some basis of V.

Solution

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