Math 520 Final Exam
Spring 2017
Brown University
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Name:

Allowed materials are pen, pencil, and straightedge. You have three hours.

Problem 1 (8 points)

The fourth and seventh columns of the matrix

$$A = \begin{bmatrix} -22 & -3 & 19 & -4 & -3 & 0 & 3 & 2 & -16 & -4 & -2 & 14 & -17 & -2 & -19 \\ 8 & 33 & -14 & 5 & 33 & -39 & 6 & -22 & -19 & -34 & -17 & 2 & -8 & 22 & 14 \\ 16 & 12 & -16 & 4 & 12 & -12 & 0 & -8 & 4 & -8 & -4 & -8 & 8 & 8 & 16 \\ -30 & -9 & 27 & -6 & -9 & 6 & 3 & 6 & -18 & 0 & 0 & 18 & -21 & -6 & -27 \\ 14 & 6 & -13 & 3 & 6 & -5 & -1 & -4 & 7 & -2 & -1 & -8 & 9 & 4 & 13 \\ 0 & -27 & 6 & -3 & -27 & 33 & -6 & 18 & 21 & 30 & 15 & -6 & 12 & -18 & -6 \end{bmatrix}$$

form a basis for the column space of A. Find a solution of the equation $A\mathbf{x} = \mathbf{b}$, where $\mathbf{b} = \begin{bmatrix} 2 \\ 17 \\ 4 \\ 0 \\ 1 \\ -15 \end{bmatrix}$.

Note: the large matrix size is supposed to be a clue that this problem is **not** at all computationally intensive. Think before calculating.

Solution

Because the fourth and seventh columns span the column space of the matrix, we can find an appropriate linear combination of them which is equal to $\mathbf{b}$. Since this linear combination has to work for the top two entries (as well as the rest of them), we can solve $\begin{bmatrix} -4 & 3 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 2 \\ 17 \end{bmatrix}$ to find that the appropriate weights are $\alpha = 1$ and $\beta = 2$. So we get

$$\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Note: this solution is not unique.

Problem 2 (5 points)

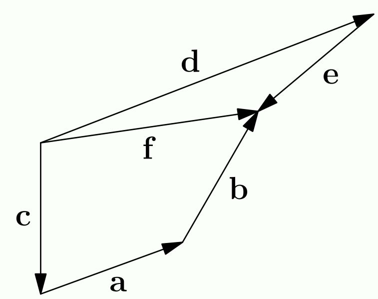
Every row operation which transforms a matrix A to a matrix A' has an *inverse* row operation which transforms A' back to A. For example, to undo scaling row j by c, we can scale row j by $\frac{1}{c}$. To undo switching rows j and k, we can just switch rows j and k again. Describe the row operation which is the inverse of "replace row j with the sum of row j and c times row k". (Hint: first try a small example to see how it goes.)

Solution

The operation that reverses the effect of adding c times row k to row j is adding $-c$ times row k to row j. This works since row k is unchanged by the original row operation, so we're subtracting off what we just added to each element of row j.

Problem 3 (5 points)

Write a vector equation relating exactly **five** of the vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}, \mathbf{f}$ in the figure below.



Solution

We have $\mathbf{a} + \mathbf{b} - \mathbf{e} - \mathbf{d} + \mathbf{c} = \mathbf{0}$.

Problem 4 (7 points)

By discussing pivots (or otherwise), explain why the solution set of a matrix equation of the form $A\mathbf{x} = \mathbf{b}$, where A is a 3×11 matrix, is either empty or infinite.

Solution

Since A has more columns than rows, there are necessarily non-pivot columns. This means that either the system is inconsistent (if the rref contains a row whose only nonzero entry is the last), or else its solution set has free variables and is therefore infinite.

Problem 5 (7 points)

Balance the following chemical equation.

$$\mathrm{Al} + \mathrm{O_2} \to \mathrm{Al_2O_3}$$

Solution

Denote by x, y, and z the number of Al atoms, the number of O_2 molecules, and the number of Al_2O_3 molecules. Then x, y, and z must satisfy the system

$$x = 2z, \quad 2y = 3z.$$

The augmented matrix of this system is

$$\begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 2 & -3 & 0 \end{bmatrix},$$

which is already in row echelon form. So we can let $z = 1$ and get $y = \frac{3}{2}$ and $x = 2$. Since fractional atoms are nonphysical, we scale up to get $z = 2$, $y = 3$, and $x = 4$. So the balanced equation is

$$4\mathrm{Al} + 3\mathrm{O_2} \to 2\mathrm{Al_2O_3}.$$

Note: positive integer multiples of the above solution are also valid solutions.

Problem 6 (6 points)

Suppose that U is an $n \times n$ orthogonal matrix (that is, its columns are orthonormal), and that A and B are invertible square matrices. Show that

$$B(AU^TB)^{-1}IAU = U^2,$$

where I denotes the $n \times n$ identity matrix. Explain your steps carefully.

Solution

We apply the fact $(XYZ)^{-1} = Z^{-1}Y^{-1}X^{-1}$ for square invertible matrices X, Y, Z to get

$$\begin{aligned} B(AU^TB)^{-1}IAU &= BB^{-1}(U^T)^{-1}A^{-1}IAU \\ &= I(U^T)^{-1}A^{-1}AU \\ &= (U^T)^{-1}U. \end{aligned}$$

Now recall that if U is orthogonal, then $U^TU = I$. If U is square, then this means that U^T and U are inverses. So the above expression reduces to U^2, as desired.

Problem 7 (8 points)

Suppose that V and W are vector spaces, $T : V \to W$ is a linear transformation, and that $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4\}$ is a basis for V such that $T(\mathbf{b}_1) = T(\mathbf{b}_2)$ and $T(\mathbf{b}_3) = T(\mathbf{b}_4)$. If $T(\mathbf{b}_1)$ is not a scalar multiple of $T(\mathbf{b}_3)$, then what is the rank of T, and what is the nullity of T? Explain your reasoning.

Solution

The range of T is contained in the span of $\{T(\mathbf{b}_1), T(\mathbf{b}_3)\}$, because

$$T(c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + c_3\mathbf{b}_3 + c_4\mathbf{b}_4) = c_1T(\mathbf{b}_1) + c_2T(\mathbf{b}_2) + c_3T(\mathbf{b}_3) + c_4T(\mathbf{b}_4) = (c_1 + c_2)T(\mathbf{b}_1) + (c_3 + c_4)T(\mathbf{b}_3).$$

Since $\{T(\mathbf{b}_1), T(\mathbf{b}_3)\}$ is not linearly dependent, it follows that the range of T is a two-dimensional subspace of W. So the rank of T is $\boxed{2}$.

The nullity of T is $4 - 2 = \boxed{2}$, by the rank-nullity theorem, since the dimension of V is 4.

Problem 8 (6 points)

Show that the zero product property *fails* for multiplication of 2×2 matrices. In other words, find two nonzero 2×2 matrices A and B which satisfy the equation $AB = 0$. (Note: the zero matrix has all entries equal to zero, and a nonzero matrix is any matrix other than the zero matrix).

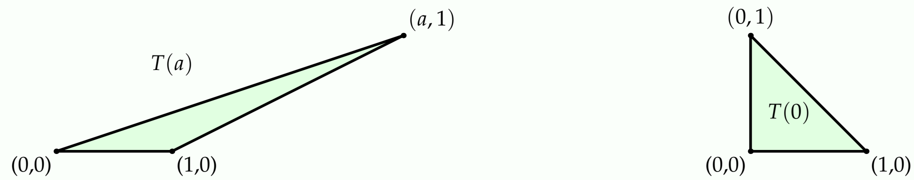
Solution

There are many possible examples. Here's one:

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Problem 9(a) (4 points)

For each real number a, consider the triangle $T(a)$ whose vertices are the origin, $(1,0)$, and $(a,1)$.



Show that the area of $T(a)$ is $\frac{1}{2}$ for all $a \in \mathbb{R}$, in two different ways:
(i) using the one-half base times height formula for the area of a triangle, and
(ii) showing that $T(a)$ is the image of $T(0)$ under the linear transformation whose matrix is $M = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$ and calculating the determinant of M (for this part, you may assume that the area of $T(0)$ is $\frac{1}{2}$). You should support your answer to (ii) with a complete verbal explanation of why the calculations you perform support the conclusion that the area of $T(a)$ is $\frac{1}{2}$.

Solution

(i) The bottom of the triangle has length 1, and the perpendicular distance from the bottom side to the top vertex is 1. So the area is $(1/2)(1)(1) = 1/2$.

(ii) The image of $T(0)$ is a triangle (since bijective linear transformations map lines to lines), and its vertices are the images of $T(0)$'s vertices under multiplication by M. We get

$$\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ 1 \end{bmatrix},$$

which are indeed the vertices of $T(a)$. Since the determinant of M is $(1)(1) - (a)(0) = 1$, and since the (absolute value of) the determinant tells us the factor by which areas are transformed, it follows that the area of $T(a)$ is $1/2$.

Problem 9(b) (3 points)

Does there exist an invertible matrix A with integer entries such that $\det(A^{-1}) = 2$? Explain.

Solution

No such matrix exists. We would have

$$\det(A) = \frac{1}{\det(A^{-1})} = \frac{1}{2},$$

which contradicts the fact that every integer-valued matrix has an integer determinant (this fact, in turn, follows from the representation of the determinant as a polynomial in the matrix entries, with coefficients in $\{\pm 1\}$).

Problem 10(a) (6 points)

Consider the set $S = \{f \in C([0,1]) : |f(x)| > 10 \text{ for all } x \in [0,1]\}$. This is the set containing all continuous functions on $[0,1]$ whose graphs either stay above the line $y = 10$ or stay below the line $y = -10$. Show that S fails to be a subspace of $C([0,1])$ on **all** three subspace criteria.

Solution

We check the three subspace conditions:

1. S does not contain the zero function, since the zero function is clearly not greater than 10 in absolute value everywhere (actually, not anywhere).
2. S is not closed under addition, because the functions $f(x) = 20 + x$ and $g(x) = -20 - 2x$ are both in S, but their sum is the function $-x$, which is in S.
3. S is not closed under scalar multiplication, beacuse $f(x) = 20$ is in S, but $\frac{1}{4}f$ is not in S.

Problem 10(b) (6 points)

A function f in $C([0,1])$ is said to be *1-periodic* if $f(0) = f(1)$. Show the set of 1-periodic functions in $C([0,1])$ is a vector subspace of $C([0,1])$.

Solution

We check the three subspace conditions.

1. The zero function is equal at 0 and 1, since its values there are 0 and 0, respectively.
2. Two functions f and g which are 1-periodic have the property that

$$(f+g)(0) = f(0) + g(0) = f(1) + g(1) = (f+g)(1),$$

so the sum $f + g$ is 1-periodic as well.

3. We have

$$(cf)(0) = cf(0) = cf(1) = (cf)(1),$$

for any 1-periodic function f, so the set of 1-periodic functions is closed under scalar multiplication as well.

Therefore, the set of 1-periodic functions is a vector subspace of $C([0,1])$.

Problem 11 (7 points)

Find the eigenvalues of

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Explain how you know that A is diagonalizable.

Solution

We calculate (expanding the determinant along the first row):

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{bmatrix} -\lambda + 3 & 0 & 0 \\ 0 & -\lambda + 2 & 1 \\ 1 & 1 & -\lambda + 1 \end{bmatrix} \\ &= (-\lambda + 3)((-\lambda + 2)(-\lambda + 1) - 1) + 1(0) \\ &= (-\lambda + 3)\left(\lambda^2 - 3\lambda + 1\right) \end{aligned}$$

finding the roots of the second factor using the quadratic formula, we get

$$\lambda \in \left\{3, \quad -\frac{\sqrt{5}}{2} + \frac{3}{2}, \quad \frac{\sqrt{5}}{2} + \frac{3}{2}\right\}.$$

Since there are three distinct eigenvalues, any eigenvectors corresponding to these eigenvalues are linearly independent. Since an $n \times n$ matrix with n linearly independent eigvectors is diagonalizable, it follows that A is diagonalizable.

Problem 12 (6 points)

Given that

$$\begin{bmatrix} 16 & -21 \\ 10 & -13 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 16 & -21 \\ 10 & -13 \end{bmatrix} \begin{bmatrix} 7 \\ 5 \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \end{bmatrix},$$

find a formula for $\begin{bmatrix} 16 & -21 \\ 10 & -13 \end{bmatrix}^n$ in terms of n. (Your answer should take the form of a 2×2 matrix each of whose entries is a formula with an n in it).

Solution

The given equations imply that the eignvectors of $A = \begin{bmatrix} 16 & -21 \\ 10 & -13 \end{bmatrix}$ are $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 7 \\ 5 \end{bmatrix}$, with eigenvalues 2 and 1, respectively. This means that $A = PDP^{-1}$, where

$$P = \begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}, \text{ and } \quad D = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}.$$

So we can calculate

$$\begin{aligned} A^n = PD^nP^{-1} &= \begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 2^n & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix} \\ &= \boxed{\begin{bmatrix} 15 \cdot 2^n - 14 & -21 \cdot 2^n + 21 \\ 10 \cdot 2^n - 10 & -14 \cdot 2^n + 15 \end{bmatrix}}. \end{aligned}$$

Problem 13(a) (5 points)

Show that the set of $n \times n$ orthogonal matrices is closed under matrix multiplication. In other words, show that if U and V are orthogonal $n \times n$ matrices, then UV is also an orthogonal matrix. (Recall that an orthogonal matrix is a matrix whose columns are orthonormal.)

Solution

Recall that U is orthogonal if and only if $U^T U = I$. Since U and V are orthogonal, we have $U^T U = I$ and $V^T V = I$. Then because $(UV)^T = V^T U^T$ (for any matrices U and V), we have

$$(UV)^T UV = V^T U^T UV = V^T I V = V^T V = I,$$

which implies that UV is orthogonal.

Problem 13(b) (5 points)

Suppose that $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ is a basis for a subspace W of $\mathbb{R}^n$, and that $\mathbf{u}$ is orthogonal to $\mathbf{b}_1$, $\mathbf{b}_2$, and $\mathbf{b}_3$. Show using properties of the dot product and the definition of *orthogonal complement* that $\mathbf{u}$ is in $W^\perp$.

Solution

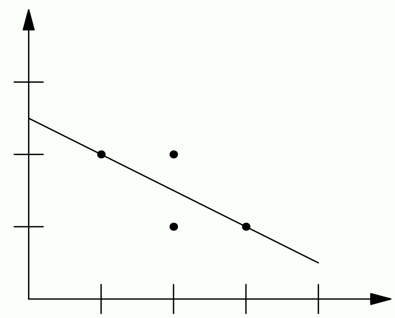
We have

$$\mathbf{u} \cdot (c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + c_3\mathbf{b}_3) = c_1\mathbf{u} \cdot \mathbf{b}_1 + c_2\mathbf{u} \cdot \mathbf{b}_2 + c_3\mathbf{u} \cdot \mathbf{b}_3 = 0 + 0 + 0 = 0$$

for any real numbers c_1, c_2, c_3. Since every vector in W can be written as a linear combination of $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$, this implies that $\mathbf{u}$ is orthogonal to every vector in W. By definition, this means that $\mathbf{u} \in W^\perp$.

Problem 14 (6 points)

Find and precisely sketch the line which does the best job of approximating these data points (here "best" means what it usually does: the sum of the squares of the lengths of the vertical segments from each of the four points to the line is as small as possible. Note that the distance between each adjacent pair of tick marks is 1).



Solution

We are looking to minimize $|A\mathbf{x} - \mathbf{b}|^2$, where $\mathbf{x} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ is a vector containing the y-intercept α and slope β, and

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}, \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 2 \\ 1 \\ 2 \\ 1 \end{bmatrix}.$$

Since $A\mathbf{x} - \mathbf{b}$ will be orthogonal to the column space of A when $\mathbf{x}$ is chosen optimally, the optimal solution satisfies

$$A^T(A\mathbf{x} - \mathbf{b}) = 0,$$

so our solution is

$$\begin{aligned} \mathbf{x} &= (A^T A)^{-1} A^T \mathbf{b} \\ &= \begin{bmatrix} 4 & 8 \\ 8 & 18 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 2 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{9}{4} & -1 \\ -1 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 6 \\ 11 \end{bmatrix} \\ &= \begin{bmatrix} \frac{5}{2} \\ -\frac{1}{2} \end{bmatrix}. \end{aligned}$$

Therefore, the line of best fit is $y = \frac{5}{2} - \frac{1}{2}x$, sketched in the figure above.

Note: this problem could actually have been solved with no calculations, because the error for the first and last data points is zero (which is as small as possible, of course), and the error for the middle two data points is as small as possible as well, since the line passes through their midpoint.

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