GUIDE TO WRITING PROOFS
SAMUEL S. WATSON
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BROWN UNIVERSITY

My number one piece of advice is **when in doubt, unpack a definition**. The majority of steps in most problems at this level are simply a matter of writing down the meaning of the last thing you wrote down. So you should have all your definitions down pat:

Definitions to know

- Set theory stuff: image and preimage of a set under a function, union/intersection
- Properties of lists of vectors: linear independence, spanning
- A linear transformation from one vector space to another, a vector subspace

Let's illustrate this through a typical example which has a lot of definition unpacking and very little else. To illustrate the thought process, the solution text is going to be really excessively verbose. **Do not** write your solutions like this.

Example 1

Suppose that V and W are vector spaces and $T : V \to W$ is a linear transformation. Suppose that U_1 and U_2 are linear subspaces of V. Show that $T(U_1 \cap U_2)$ is a linear subspace of W.

Solution

Maybe we feel stuck right off the bat. So we ask ourselves: is there a definition to unpack? We want to show that "$T(U_1 \cap U_2)$ is a linear subspace of W". What does that mean? We need to know what a linear subspace is to even make sense of that sentence. So let's apply the definition of a linear subspace. "$T(U_1 \cap U_2)$ is a linear subspace of W" is shorthand for:

- $\mathbf{0} \in T(U_1 \cap U_2)$,
- $\mathbf{v}, \mathbf{w} \in T(U_1 \cap U_2)$ implies that $\mathbf{v} + \mathbf{w} \in T(U_1 \cap U_2)$, and
- $\mathbf{v} \in T(U_1 \cap U_2)$ implies that $c\mathbf{v} \in T(U_1 \cap U_2)$ for all $c \in \mathbb{R}$

Note that *we have not done any hard thinking here*. We literally just copy-pasted $T(U_1 \cap U_2)$ into the definition of a linear subspace. So now we need to check those three bullet points.

Why is $\mathbf{0} \in T(U_1 \cap U_2)$? Maybe we feel stuck again. So again we ask ourselves, what does this mean? In other words, what does it mean for a particular vector to be in the image of some set $U_1 \cap U_2$ under a given linear transformation T? By definition, it means that there is some vector in $U_1 \cap U_2$ that maps to it. So here we have to do a little non-definitional thinking: what is a vector that maps to $\mathbf{0}$ in W? The zero vector in V is a natural choice to consider here, since we know that linear transformations map $\mathbf{0}$ to $\mathbf{0}$. So can we be sure that $\mathbf{0} \in U_1 \cap U_2$? Again, we ask ourselves what that means. It means that $\mathbf{0} \in U_1$ and $\mathbf{0} \in U_2$. How can we be sure of that? Well, U_1 and U_2 are subspaces of V, so by definition they must contain the zero vector. So indeed $\mathbf{0} \in T(U_1 \cap U_2)$.

Next we check the second bullet point. We want to prove an "A implies B" type of statement, and to tackle those we assume A and see if we can do some work to conclude B from that. So let's assume $\mathbf{v}, \mathbf{w} \in T(U_1 \cap U_2)$. What does that mean? Well, $\mathbf{v} \in T(U_1 \cap U_2)$ means that there is a vector $\mathbf{x} \in U_1 \cap U_2$ which maps to $\mathbf{v}$ under T. Similarly, $\mathbf{w} \in T(U_1 \cap U_2)$ means that there is a vector $\mathbf{y} \in U_1 \cap U_2$ which maps to $\mathbf{v}$ under T. OK, so now we have these vectors $\mathbf{x}$ and $\mathbf{y}$ in $U_1 \cap U_2$. What should we do with them? Maybe it's difficult to see what to do here, so instead we look at the thing we're trying to show and see if we can work that down to a simpler form.

We want to conclude that $\mathbf{v} + \mathbf{w} \in T(U_1 \cap U_2)$, which is the same thing as saying that there is some $\mathbf{z} \in U_1 \cap U_2$ which maps to $\mathbf{v} + \mathbf{w}$. So what can we do with $\mathbf{x}$ and $\mathbf{y}$? Adding them feels like a natural thing to do. Is it the case that if we say $\mathbf{z} = \mathbf{x} + \mathbf{y}$ that $T(\mathbf{z})$ is indeed equal to $\mathbf{v} + \mathbf{w}$? Yes, by linearity: $T(\mathbf{x} + \mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y}) = \mathbf{v} + \mathbf{w}$. But don't forget that we also need to check that $\mathbf{x} + \mathbf{y}$ is in $U_1 \cap U_2$.

So what does it mean to say that $\mathbf{x} + \mathbf{y}$ is in $U_1 \cap U_2$? It means that $\mathbf{x} + \mathbf{y}$ is in U_1 and also in U_2. But we know $\mathbf{x} + \mathbf{y}$ is in U_1 since U_1 is a subspace. Similarly, we know it's in U_2. So that's it for bullet point 2! We have figured out that $T(U_1 \cap U_2)$ is closed under vector addition.

Closure under scalar multiplication (bullet point 3) works almost identically to closure under vector addition, so we omit it.