A CRASH COURSE IN SET THEORY
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1 Sets

A **set** is a collection of elements. These elements can be numbers, points, shapes, vectors, other sets, whatever. For example,

$$A = \{1, 4, 9\}$$

is the set consisting of the single-digit perfect squares. The main thing you can do with a set is check whether a particular element is in it. For example, we say that $1 \in A$ (read "1 is an element of A"), while $2 \notin A$ ("2 is not an element of A").

Some sets with standard and specially typeset names include

- $\mathbb{R}$, the set of real numbers,
- $\mathbb{Q}$, the set of rational numbers,
- $\mathbb{Z}$, the set of integers, and
- $\mathbb{N}$, the set of natural numbers.

We say that $A \subset B$ (read "A is a **subset** of B") if every element of A is an element of B. For example,

$$\{1, 4, 9\} \subset \{1, 2, 3, 4, 9, 10\}.$$

We say that two sets A and B are **equal** if $A \subset B$ and $B \subset A$. Note that

$$\{1, 1, 2\} = \{1, 2\} = \{2, 1\}.$$

since each element of each set is in the others. Thus we can see that sets "don't care" about repeated elements or order. All that matters is what is in and what is not. It is customary to write sets with repeats omitted, for clarity.

We write $A \cap B$, the **intersection** of A and B, for the set of all the elements that are in both A and B. So, for example,

$$\{1, 4, 9\} \cap \{x \in \mathbb{R} : x^2 > 15\} = \{4, 9\}.$$

That second set on the left-hand side, which is written in *set-builder* notation, is read as "the set of all real numbers x such that the square of x is greater than 15".

We write $A \cup B$, the **union** of A and B, for the set of all the elements that are in either A or B. So, for example,

$$\{1, 4, 9\} \cup \{1, 9, 25\} = \{1, 4, 9, 25\}.$$

2 Functions

If A and B are sets, then a function $f : A \to B$ is a rule that assigns a single element of B to each element of A. The set A is called the **domain** of f and B is called the **codomain** of f. Given a subset A' of A, we define the **image** $f(A')$ to be

$$f(A') = \{b \in B : \text{there exists } a \in A' \text{ so that } f(a) = b\}.$$

This is the set of all elements of B that get mapped to from some element of A'. The **range** of f is defined to be the set $f(A)$, which contains all the elements of B that get mapped to at least once.

Similarly, if $B' \subset B$, then the **preimage** $f^{-1}(B')$ of B' is defined by

$$f^{-1}(B') = \{a \in A : f(a) \in B'\}.$$

This is the subset of A consisting of every element of A that maps to some element of B'.

A function f is **injective** if no two elements in the domain map to the same element in the codomain; in other words if $f(a) = f(a')$ implies $a = a'$.

A function f is **surjective** if the range of f is equal to the codomain of f; in other words, if $b \in B$ implies that there exists $a \in A$ with $f(a) = b$.

A function f is **bijective** if it is both injective and surjective. This means that for every $b \in B$, there is exactly one $a \in A$ such that $f(a) \in b$. If f is bijective, then the function from B to A that maps $b \in B$ to the element $a \in A$ that satisfies $f(a) = b$ is called the **inverse** of f.

If the rule defining a function is sufficiently simple, we can describe the function using *anonymous function notation*. For example, $x \in \mathbb{R} \mapsto x^2 \in \mathbb{R}$, or $x \mapsto x^2$ for short, is the squaring function from $\mathbb{R}$ to $\mathbb{R}$. Note that bar on the left edge of the arrow, which distinguishes the arrow in anonymous function notation from the arrow between the domain and codomain of a named function.

3 Lists

In linear algebra, we sometimes want to use a structure that *can* account for repetitions and order. For example, if $\mathbf{v}$ is a vector in a vector space V, we want to be able to say that $\{\mathbf{v}, \mathbf{v}\}$ is linearly dependent. However, $\{\mathbf{v}, \mathbf{v}\}$ *as a set* is equal to the set $\{\mathbf{v}\}$, as we saw above.

For this reason, we use the notion of a *list*. A **list** of elements of V, formally speaking, is a function from $\{1, 2, \ldots, n\}$ to V, where n is some positive integer. However, we typically skip the function formalism and just write out the elements in order:

$$\{\mathbf{u}, \mathbf{u}, \mathbf{v}\}$$

is a list whose first element is $\mathbf{u}$, whose second element is $\mathbf{u}$, and whose third element is $\mathbf{v}$.