Brown University
Problem Set 6
Instructor: Samuel S. Watson
Due: 27 October 2017

Print out these pages, including the additional space at the end, and complete the problems by hand. Then use Gradescope to scan and upload the entire packet by 18:00 on the due date.

Problem 1

Find the directions in which the directional derivative of $f(x, y) = ye^{-xy}$ at the point $(0, 2)$ has value 1.

Solution

Problem 2

Find the derivative with respect to t of the function $g(t) = t^t$ by writing the function as $f(x(t), y(t))$ where $f(x, y) = x^y$ and $x(t) = t$ and $y(t) = t$.

Solution

Problem 3

In this problem, we explore—in the context of the chain rule—what happens when we are careful with the errors associated with linear approximation, instead of just throwing them away and repeating the word "roughly".

For simplicity, suppose that $\mathbf{r}$ is a differentiable path satisfying $\mathbf{r}(0) = (0,0)$, and suppose that $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable at the origin with $f(0,0) = 0$. We will show that $(f \circ \mathbf{r})'(0) = (\nabla f)(0,0) \cdot \mathbf{r}'(0)$.

(a) Suppose that L is the linearization of f at the origin. Consider the facts

$$f(x,y) = L(x,y) + R(x,y)|\langle x,y\rangle|, \tag{3.1}$$

where $R(x,y) \to 0$ as $(x,y) \to (0,0)$, and

$$\mathbf{r}(t) = t\,\mathbf{r}'(0) + t\,\mathbf{E}(t), \tag{3.2}$$

where $|\mathbf{E}(t)| \to 0$ as $t \to 0$. Fill in the blank: (3.1) follows from the fact that f is ________________ , by definition. Equation (3.2) follows from the fact that $\mathbf{r}$ is ________________.

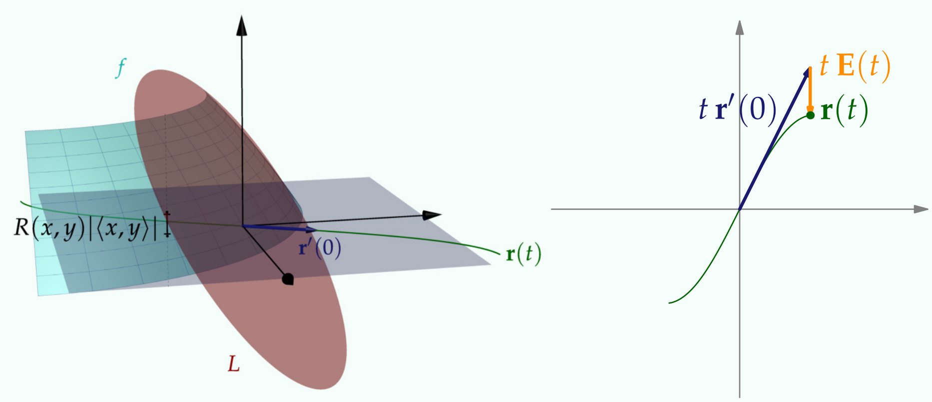
(b) Substitute one of the equations from (a) into the other to show that

$$\frac{f(\mathbf{r}(h))}{h} = L(\mathbf{r}'(0)) + L(\mathbf{E}(h)) \pm R(h\,\mathbf{r}'(0) + h\,\mathbf{E}(h))|\mathbf{r}'(0) + \mathbf{E}(h)|. \tag{3.3}$$

You will want to make use of the fact that L is linear in the sense that $L(\alpha\mathbf{v} + \beta\mathbf{w}) = \alpha L(\mathbf{v}) + \beta L(\mathbf{w})$ for all scalars α and β and vectors $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^2$.

(c) For each of the terms on the right-hand side of (3.3) except the first, explain why it converges to zero as $h \to 0$.

(d) Take the limit of both sides of (3.3) as $h \to 0$ and thereby establish the chain rule.



Solution

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