BROWN UNIVERSITY
Math 0200 Exam I
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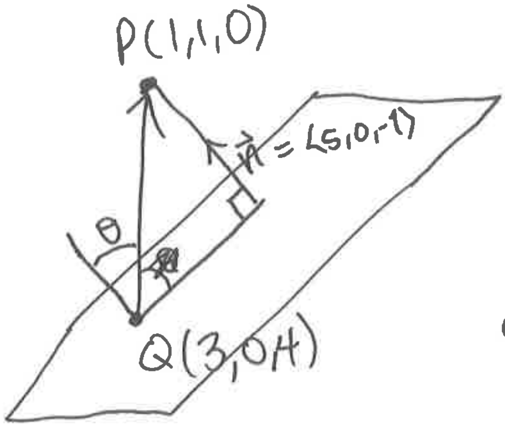
Key

1. (10 points) Find the point on the plane $5x - z = 11$ which is closest to the point $(1, 1, 0)$.



$$\overrightarrow{QP} = \langle -2, 1, -4\rangle$$
$$\vec{n} = \langle 5, 0, -1\rangle$$

$$\text{distance to } P = |\overrightarrow{QP}||\cos\theta| = \frac{|\overrightarrow{QP}\cdot\vec{n}|}{|\vec{n}|} = \frac{|\langle -2,1,-4\rangle\cdot\langle 5,0,-1\rangle|}{\sqrt{26}} = \frac{|-10+0+4|}{\sqrt{26}} = \frac{6}{\sqrt{26}}$$

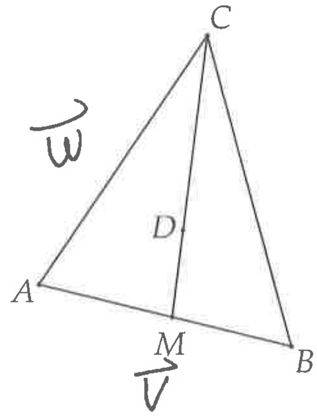
So the vector from P to the foot of the perpendicular from P is equal to

$$\left(\frac{6}{\sqrt{26}}\right)\frac{\vec{n}}{|\vec{n}|} = \frac{6\langle 5,0,-1\rangle}{(\sqrt{26})^2} = \frac{3}{13}\langle 5,0,-1\rangle = \left\langle \frac{15}{13}, 0, \frac{-3}{13}\right\rangle$$

So the foot equals $\langle 1,1,0\rangle + \left\langle\frac{15}{13}, 0, \frac{-3}{13}\right\rangle = \boxed{\left\langle \frac{28}{13}, 1, \frac{-3}{13}\right\rangle}$

2. A *median* of a triangle is a line segment connecting a midpoint of one of the sides with the opposite vertex. In this problem, we will use vector addition, subtract, and multiplication to show that there is a point, called the *centroid*, which lies on all three medians. Consider a triangle with vertices A, B, and C, and define $\mathbf{v} = \overrightarrow{AB}$ and $\mathbf{w} = \overrightarrow{AC}$.

Let's call D the point which is two-thirds of the way from C to the midpoint of $\overline{AB}$. In other words, D is on the line through C and the midpoint M of $\overline{AB}$, and it's twice as far from C as it is from M.



(a) (3 points) Write a simple expression for $\overrightarrow{CD}$ in terms of the vector $\overrightarrow{CM}$.

$$\overrightarrow{CD} = \frac{2}{3}\overrightarrow{CM}$$

(b) (3 points) Express $\overrightarrow{CM}$ in terms of $\mathbf{v}$ and $\mathbf{w}$.

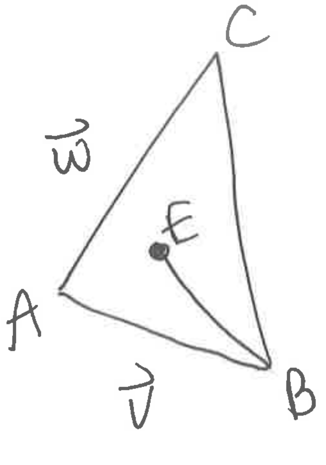
$$\overrightarrow{CM} = \frac{1}{2}\vec{v} - \vec{w}$$

(c) (5 points) Using (a) and (b), express $\overrightarrow{AD}$ in terms of $\mathbf{v}$ and $\mathbf{w}$.

$$\overrightarrow{AD} = \vec{w} + \overrightarrow{CD} = \vec{w} + \frac{2}{3}\left(\frac{1}{2}\vec{v} - \vec{w}\right)$$

$$= \frac{1}{3}(\vec{v} + \vec{w})$$

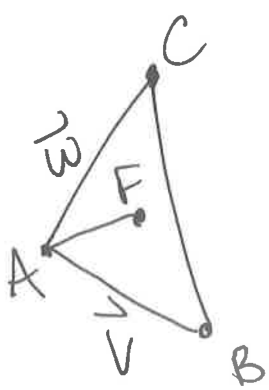
(b) (2 points) Let's call E the point which is two-thirds of the way from B to the midpoint of $\overline{AC}$. Express $\overrightarrow{AE}$ in terms of **v** and **w**.



$$\vec{AE} = \vec{v} + \frac{2}{3}\left(\frac{1}{2}\vec{w} - \vec{v}\right)$$

$$= \frac{1}{3}(\vec{v} + \vec{w})$$

(c) (2 points) Let's call F the point which is two-thirds of the way from A to the midpoint of $\overline{BC}$. Express $\overrightarrow{AF}$ in terms of **v** and **w**.

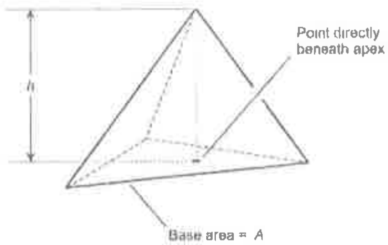


$$\vec{AF} = \frac{2}{3}\left(\frac{1}{2}(\vec{v} + \vec{w})\right)$$

$$= \frac{1}{3}(\vec{v} + \vec{w})$$

(You will see that $\overrightarrow{AD} = \overrightarrow{AE} = \overrightarrow{AF}$. This shows that $D = E = F$.)

3. Recall that the volume of a tetrahedron is equal to $1/3$ times the area A of its base times its height h (see the figure below).



The four planes $x+y=1$, $x+z=1$, $y+z=1$, and $x+y+z=1$ divide $\mathbb{R}^3$ up into several regions, one of which is a tetrahedron T.

(a) (6 points) Find the vertices of T.

$(1,0,0)$, $(0,1,0)$, $(0,0,1)$, $(\frac{1}{2},\frac{1}{2},\frac{1}{2})$

(solve a system of three of these equations, each of four possible ways).

(b) (6 points) Find the area of any one the faces of T.

the $(1,0,0),(0,1,0),(0,0,1)$ face:

$$\left|\frac{1}{2}\left|\begin{matrix}\vec{i} & \vec{j} & \vec{k}\\ 0 & 1 & -1\\ -1 & 1 & 0\end{matrix}\right|\right| = \left|\frac{1}{2}(\vec{i}+\vec{j}+\vec{k})\right| = \boxed{\frac{\sqrt{3}}{2}}$$

the other faces:

$$\frac{1}{2}\left|\left|\begin{matrix}\vec{i} & \vec{j} & \vec{k}\\ 1/2 & -1/2 & -1/2\\ 1 & 0 & -1\end{matrix}\right|\right|$$

$$= \frac{1}{2}\left|\left\langle \frac{1}{2}, 0, 1/2\right\rangle\right|$$

$$= \frac{1}{2}\sqrt{\frac{1}{4}+\frac{1}{4}}$$

$$= \frac{1}{2\sqrt{2}} = \boxed{\frac{\sqrt{2}}{4}}$$

(c) (8 points) Find the distance from the face you found in part (b) to the vertex which is not contained in that face. Use your answer and the answer to (b) to find the volume of T.

the $(1,0,0),(0,1,0),(0,0,1)$ face

(*) point: $(\frac{1}{2},\frac{1}{2},\frac{1}{2}) = P$

(*) plane through $(1,0,0) = Q$ with normal $\langle 1,1,1\rangle$

$$\text{distance} = \frac{|\vec{n}\cdot\overrightarrow{QP}|}{|\vec{n}|}$$

$$= \frac{|\langle 1,1,1\rangle\cdot\langle -\frac{1}{2}, +\frac{1}{2}, \frac{1}{2}\rangle|}{\sqrt{3}}$$

$$= \frac{1}{2\sqrt{3}}$$

$$\text{volume} = \frac{1}{3}\cdot\frac{\sqrt{3}}{2}\cdot\frac{1}{2\sqrt{3}} = \boxed{\frac{1}{12}}$$

the other faces

point: $(1,0,0) = P$

plane through $(0,1,0) = Q$

with normal ~~$\langle 1,1,0\rangle$~~ $\langle 1,0,1\rangle$

$$\text{distance} = \frac{\vec{n}\cdot\overrightarrow{QP}}{|\vec{n}|}$$

$$= \frac{\langle 1,0,1\rangle\cdot\langle 1,-1,0\rangle}{|\vec{n}|}$$

$$= \frac{1}{\sqrt{2}}$$

volume

$$= \frac{1}{3}\cdot\frac{1}{\sqrt{2}}\cdot\frac{\sqrt{2}}{4} = \boxed{\frac{1}{12}}$$

4. A particle moves in space so that its location after t seconds is given by the position vector

$$\mathbf{r}(t) = \langle t\cos(2\pi t/3), t\sin(2\pi t/3), t\rangle$$

until the time when particle reaches the sphere $x^2 + y^2 + z^2 = 2$. After that time, the particle moves in a straight line, directly away from the origin at a rate of one unit per second.

(a) (6 points) Find the coordinates of the particle's location at time $t = 5$.

$$|\vec{r}(t)|^2 = t^2\cos^2(2\pi t/3) + t^2\sin^2\left(\frac{2\pi t}{3}\right) + t^2$$

$$= 2t^2$$

So $|\vec{r}(t)|^2 = 2$ when $t = 1$. At that time, its location is

$$\vec{r}(1) = \left\langle \cos\left(\frac{2\pi}{3}\right), \sin\left(\frac{2\pi}{3}\right), t\right\rangle$$

$$= \left\langle -\frac{1}{2}, \frac{\sqrt{3}}{2}, 1\right\rangle$$

After 4 more seconds, its distance from the origin will have increased by a factor of $\underbrace{\frac{\sqrt{2}+4}{\sqrt{2}}}_{=1+2\sqrt{2}}$, from $\sqrt{2}$ to $\sqrt{2}+4$, so

$$\vec{r}(5) = \left\langle -\frac{1}{2}(1+2\sqrt{2}), \frac{\sqrt{3}}{2}(1+2\sqrt{2}), 1+2\sqrt{2}\right\rangle$$

(b) (6 points) Find the particle's velocity at time $t = 3/4$.

$$\vec{r}\,'(t) = \left\langle \cos(2\pi t/3) + t\left(-\sin\left(\frac{2\pi t}{3}\right)\left(\frac{2\pi}{3}\right)\right),\ \sin\left(\frac{2\pi t}{3}\right) + t\cos\left(\frac{2\pi t}{3}\right)\left(\frac{2\pi}{3}\right),\ 1 \right\rangle$$

$$= \left\langle -\frac{\pi}{2}, 1, 1 \right\rangle$$

(c) (6 points) Find the arclength of the particle's path over the period from $t = 0$ to $t = 5$.

$$\text{arclength} = 4 + \int_0^1 \left(\left[\cos\left(\frac{2\pi t}{3}\right) - t\sin\left(\frac{2\pi t}{3}\right)\left(\frac{2\pi}{3}\right)\right]^2 + \left[\sin\left(\frac{2\pi t}{3}\right) + \frac{2\pi t}{3}\cos\left(\frac{2\pi t}{3}\right)\right]^2 + 1^2 \right)^{1/2} dt$$

$$= 4 + \int_0^1 \sqrt{2 + \frac{4\pi^2}{9}t^2}\, dt$$

5. Consider the function f defined by

$$f(x,y) = \begin{cases} x^2 & \text{if } y = 0 \\ y^2 & \text{if } x = 0 \\ 1 & \text{otherwise.} \end{cases}$$

(a) (8 points) Explain carefully why $\frac{\partial f}{\partial x}(0,0) = 0$ and $\frac{\partial f}{\partial y}(0,0) = 0$.

$\frac{\partial f}{\partial x}(0,0)$ = the derivative at 0 of f restricted to the line $y=0$

$= \frac{d}{dx}[x^2]\Big|_{x=0} = 2x\Big|_{x=0} = 0$

$\frac{\partial f}{\partial y}(0,0) = 0$ for the same reason.

(b) (8 points) Explain why f is not differentiable at $(x,y) = (0,0)$. (Hint: is f continuous at $(0,0)$?)

f is not continuous at (0,0) because

$f(0,0) = 0$ but $\lim_{x \to 0} f(x,x) = 1 \neq 0$,

for instance.

6. (8 points) Find

$$\frac{\partial}{\partial x}\frac{\partial}{\partial x}\frac{\partial}{\partial y}\left[x(e^{\sqrt{\ln y}+y^{10}e^{y}-11\sin(\sqrt{y\tan y})}+x^6)\right]$$

clairaut's theorem

$$= \frac{\partial}{\partial y}\frac{\partial}{\partial x}\frac{\partial}{\partial x}\left(\text{ same } \right)$$

$$= \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\left(7x^6\right)\right)$$

$$= \frac{\partial}{\partial y}\left(42x^5\right) = \boxed{0}$$

7. Suppose that f is a function from $\mathbb{R}^2$ to $\mathbb{R}$, some of whose values are shown in the table below.

		x						
		-3	-2	-1	0	1	2	3
	3	36	31	28	27	28	31	36
	2	17	12	9	8	9	12	17
	1	10	5	2	1	2	5	10
y	0	9	4	1	0	1	4	9
	-1	8	3	0	-1	0	3	8
	-2	1	-4	-7	-8	-7	-4	1
	-3	-18	-23	-26	-27	-26	-23	-18

(a) (4 points) Estimate $f_x(1.5, 3)$.

$$\approx \frac{\text{rise}}{\text{run}} = \frac{31-28}{1} = 3$$

(b) (4 points) Determine (with explanation) the sign of $f_{yy}(-2, 1)$.

the y slope goes from 1 to 7 from below to above $(-2,1)$, so $f_{yy} > 0$ at $(-2,1)$.

(c) (5 points) In fact, for the function f used to generate the above table, there exists a constant c such that $f_{xy}(x, y) = c$ for all $(x, y) \in \mathbb{R}^2$. Find c, and explain your reasoning.

Look at any four entries in a square shape. the x-slope stays the same as we move up one row, so $f_{xy} = 0$.