MATH 19 PROBLEM SET 8
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1 Suppose that $(a_n)_{n=1}^{\infty}$ is a sequence of positive numbers for which $\frac{a_{n+1}}{a_n}$ converges to $\frac{1}{2}$ as $n \to \infty$.

(a) What can you conclude about the convergence or divergence of $\sum_{n=1}^{\infty} a_n$?

(b) What can you conclude about the convergence or divergence of $(a_n)_{n=1}^{\infty}$?

Explain why your conclusions are correct.

2 Determine the convergence of divergence of each of the following series.

(a) $\frac{1^3}{10} + \frac{2^3}{100} + \frac{3^3}{1000} + \frac{4^3}{10000} + \cdots$

(b) $\sum_{n=1}^{\infty} \frac{n!}{100^n}$

(c) $\sum_{n=1}^{\infty} \frac{2^{n^2}}{n!}$

(d) $\sum_{n=1}^{\infty} \frac{n!}{n^n}$

3 The Indian mathematician Srinivasa Ramanujan (about whom there is a pretty decent movie that was in cinemas earlier this year: *The Man Who Knew Infinity*) discovered the following amazing formula:

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{n=0}^{\infty} \frac{(4n)!(1103 + 26390n)}{(n!)^4 396^{4n}}$$

Verify that this series is indeed convergent.

4 (IMPORTANT) One test we did not cover in class is called the **limit comparison** test, which says:

> If $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are sequences of positive numbers and $\frac{a_n}{b_n}$ converges to a positive number as $n \to \infty$, then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ either both converge or both diverge.

For example, $\sum_{n=1}^{\infty} \arctan\left(\frac{1}{n}\right)$ diverges since $\frac{\arctan(1/n)}{1/n}$ converges to 1 as $n \to \infty$, and the series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. The beauty of the limit comparison test, compared to the regular comparison test, is that it's less sensitive to things like plus/minus signs. Redo this problem from the last homework:

> Determine whether $\sum_{n=1}^{\infty} \frac{1}{3^n - 2^n}$ converges.

using the limit comparison test with $a_n = \frac{1}{3^n - 2^n}$ and $b_n = \frac{1}{3^n}$. Note: this test is particularly handy when (i) you want to throw away terms which are being added to terms that dominate them, or (ii) you're dealing with a weird function you want to get rid of, like arctan in the above example.

5 Indicate whether each series is divergent, conditionally convergent, or absolutely convergent. Hint: for the last one, you're going to need the limit comparision test from the previous exercise.

(a) $\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} - \frac{1}{\sqrt{5}} + \cdots$

(b) $\sum_{n=1}^{\infty} \cos(\pi n) n e^{-n}$

(c) $\sum_{n=1}^{\infty} (-1)^n e^{2/n}$

(d) $\sum_{n=1}^{\infty} (-1)^n \sin(\pi/n)$

6 (a) Substitute $t = 1$ in the equation marked (11.3) in the notes to find an alternating infinite series which sums to $\ln 2$.

(b) Let S_n be the nth partial sum of the series from (a). Using a calculator or otherwise, determine for $n \in \{1, 2, 3, 4, 5, 6\}$ whether S_n is an overestimate or an underestimate of $\ln 2$. Describe the pattern in your sequence of answers, and explain why the pattern must continue for all n. (Hint: you might find some inspiration in the text following the statement of Theorem 11.10)

7 Find the linear and quadratic approximations of each function at the specified point.

(a) $f(x) = \frac{1}{1+x}$ at $x = 0$

(b) $f(x) = \ln x$ at $x = 1$

8 Calculate the quadratic approximation $Q(x)$ of $f(x) = \cos x$ centered at $x = 0$.

(a) Use a calculator to show that $Q(0.1)$ is within 10^{-5} of $\cos(0.1)$.

(b) Find $\lim_{x\to 0} \frac{1-Q(x)}{x^2}$. Use L'Hospital's rule to verify that this limit is equal to $\lim_{x\to 0} \frac{1-\cos(x)}{x^2}$.

9 Determine the convergence or divergence of $\sum_{n=0}^{\infty} (2^{1/n} - 1)^n$ (this one is less complicated than it seems).

10 A surprising feature of conditionally convergent series is that the terms can be rearranged to give a *different sum*. For example,

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots = 0.693147180\ldots$$

Show that the sum of the series obtained by rearranging the terms to get a $++-$ pattern is greater than $0.693147180\ldots$:

$$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \cdots > 0.693147180\ldots$$

Hint: Write the "$++-$" sum as $\sum_{k=0}^{\infty} \left(\frac{1}{4k+1} + \frac{1}{4k+3} - \frac{1}{2k+2} \right)$ and show that the kth term of this series is positive for all k. Then show that the first term is already bigger than $0.693147180\ldots$ (you won't need a calculator for this).