MATH 19 PROBLEM SET 7
FALL 2016
BROWN UNIVERSITY
SAMUEL S. WATSON

1 A weight is placed on a spring. The spring is stretched from rest and then released. Every second, a data point about the position of the spring is collected. This data is collected in the sequence $(p_n)_{n=0}^{\infty}$. The experimental data shows that $p_n = e^{-n/3}(\sin(4n) + 3\cos(4n))$ approximately. After many seconds, does the position of the spring approach a certain value? What does this data suggest about the long term behaviour of the spring?

Note: the following code (which can be run at sagecell.sagemath.org) can help you plot this sequence.

```
line([(n,exp(-n/3)*(sin(n)+3*cos(n))) for n in range(20)])
```

2 Suppose that $(a_n)_{n=1}^{\infty}$ is a sequence of real numbers with the property that $\sum_{n=1}^{\infty} a_n = 15$. Let's define the partial sums $S_k = \sum_{n=1}^{k} a_n$. Find the values of

$$\lim_{n\to\infty} a_n \quad \text{and} \quad \lim_{k\to\infty} S_k.$$

3 A friend explains to you that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ converges, because the amount $\frac{1}{\sqrt{n}}$ that you add with each new term gets smaller and smaller as $n \to \infty$. Is your friend correct in saying that the series passes the nth term test? Is your friend correct in saying that the series converges?

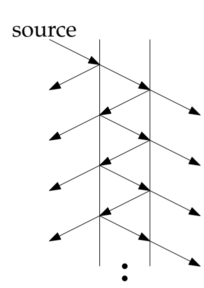
4 (True or False) For each statement, explain why the statement is true, or show that the statement is false by providing a counterexample.

(i) If f is a continuous function such that $\int_1^{\infty} f(x)\,dx$ converges, then $\sum_{n=1}^{\infty} f(n)$ converges too.

(ii) If $\sum_{n=1}^{\infty} a_n$ converges and $0 \le a_n \le b_n$, then $\sum_{n=1}^{\infty} b_n$ converges.

(iii) If $\sum_{n=1}^{\infty} a_n$ is a geometric series, then it converges.

5 A beam of light is directed at a pair of parallel glass panes. Each time the beam strikes a glass surface, a fraction p of the light is transmitted through the glass, and the remaining fraction $1-p$ is reflected. What is the total fraction of the original light beam that passes through both panes (in other words, the sum over all the arrows exiting the diagram to the right)? What is the total fraction which is reflected (i.e., exiting to the left)? What is the total fraction which remains between the panes forever? Express your answers in terms of p.



6 Determine whether each of the following series converges or diverges.

(a) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^2+4}}$

(c) $\displaystyle\sum_{n=1}^{\infty} ne^{-n}$ (d) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1.001}}$

7 Use the comparison test to determine whether each of the following series converges or diverges.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2-1}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+n+e^{2n}}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n-2^n}$ (d) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(1/n)}{n^2}$

8 Find $\displaystyle\sum_{k=1}^{\infty} k2^{-k}$ by considering

$$\begin{array}{ccccccc} \frac{1}{2} & + & \frac{1}{4} & + & \frac{1}{8} & + & \cdots \\ & + & \frac{1}{4} & + & \frac{1}{8} & + & \cdots \\ & & & + & \frac{1}{8} & + & \cdots \\ & & & & & \vdots & \end{array}$$

and adding these numbers up (i) by rows and (ii) by columns (one way you'll get an actual number, and the other way you'll get $\sum_{k=1}^{\infty} k2^{-k}$).

9 The geometric series formula

$$a + ar + ar^2 + \cdots + ar^n = \frac{ar^{n+1} - a}{r-1}$$

holds for any *complex* numbers a and r, as long as $r \neq 1$. Find an closed-form expression (that is, an expression without an ellipsis or summation symbol) for

$$\sin\theta + \sin 2\theta + \sin 3\theta + \cdots + \sin n\theta$$

by (i) writing it as the imaginary part of a sum of exponentials, using Euler's formula, (ii) simplifying the sum of exponentials using the geometric series formula, (iii) multiplying the numerator and denominator of the simplified expression by $e^{-i\theta/2}$, (iv) switching the representation of the denominator back to trig functions using Euler's formula again, and (v) taking the imaginary part of the resulting expression.

10 A prime number is a number with exactly two distinct factors: one and itself. The first several primes are 2, 3, 5, 7, 11, 13, 17, 19, ...

A subsequence of a sequence $(a_n)_{n=0}^{\infty}$ is a sequence $(b_k)_{k=0}^{\infty}$ such that $b_k = a_{n_k}$, where $n_0 < n_1 < n_2 < \ldots$. In other words, $(b_k)_{k=0}^{\infty}$ can be obtained from $(a_n)_{n=0}^{\infty}$ by deleting certain elements of $(a_n)_{n=0}^{\infty}$. For example the sequence 1,0,1,0,1,0,1,. . . has the subsequence 1,1,1,1,1,. . . and the subsequence 0,0,0,0,0,0,. . .. The sequence 1,2,3,4,5,6,. . . has the increasing even numbers as a subsequence, the sequence 2,3,4,5,6,. . . as a subsequence, and the sequence of increasing powers of 2 as a subsequence (and many more!).

Let $(a_n)_{n=0}^{\infty}$ be a sequence. For every prime number p we define B_p to be the subsequence $(b_k)_{k=0}^{\infty}$ such that $b_k = a_{pk}$. So, for example, $B_2 = a_0, a_2, a_4, a_6, \ldots$ and $B_7 = a_0, a_7, a_{14}, a_{21}, \ldots$. Is it true that if B_p converges for every prime number p, then the original sequence $(a_n)_{n=0}^{\infty}$ must also converge? If you think it is true, give an argument supporting your claim. If you think it's false, provide a counterexample.