MATH 19

Problem Set 5

Solutions.

1. (a) linear. Non-homogeneous. Constant coefficients.

(b). linear. homogeneous. Non-Constant coefficients.

(c) linear. homogeneous. Constant coefficients.

(d). linear. homogeneous. non-constant coefficients.

(e). linear. homogeneous. Constant coefficients.

(f). nonlinear.

2. Since $(e^{-x})' = -e^{-x}$, we can guess $f(x) = A + Be^{-x}$

$f'(x) = -Be^{-x}$. $f'(x) + f(x) = A$. Let $A = 1$, B be any constant that is not zero.

~~then~~ Then $\boxed{f(x) = 1 + Be^{-x}}$ satisfies the equation.

3. $f = e^{\lambda x}$.

(a). $f'' + 4f' + 3f = 0. \Rightarrow \lambda^2 + 4\lambda + 3 = 0$. $\lambda = -1$ or -3. $\boxed{f(x) = Ae^{-x} + Be^{-3x}}$

(b). $f'' - 6f + 13 = 0. \Rightarrow \lambda^2 - 6\lambda + 13 = 0$ $\lambda = 3 \pm 2i$. $\boxed{f(x) = Ae^{3x}\cos 2x + Be^{3x}\sin 2x}$

(c). $f'' + f = 0. \Rightarrow \lambda^2 + 1 = 0$. $\lambda = \pm i$. $\boxed{f(x) = A\cos x + B\sin x}$

(d). $f'''' = f \Rightarrow \lambda^4 - 1 = 0$. $\lambda = 1, i, -i,$ or -1

$\therefore \boxed{f(x) = Ae^{x} + B\cos x + C\sin x + De^{-x}}$

4.

4. $m=1$, $d=3$, $k=2$. $y''(t)+3y'(t)+2=0$. $\lambda^2+3\lambda+2=0$. $\lambda=-1$ or -2.

The solution is $\boxed{y(t)=Ae^{-t}+Be^{-2t}}$ which is not oscillatory, just decaying.

This makes sense because d is large here, meaning the damping force is large.

5. We can consider an equation with $\lambda=1\pm2i$ as the roots of its characteristic equation, so we can take $\boxed{f''(x)-2f'(x)+5f(x)=0.}$

whose characteristic equation is $\lambda^2-2\lambda+5=0$.

6. If $g(x)$, $h(x)$ satisfies

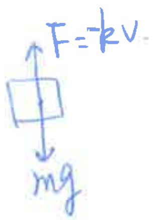
$$a g''(x)+b g'(x)+c g(x)=0. \quad a h''(x)+b h'(x)+c h(x)=0.$$

Multiply with A and B respectively, summing these two equalities, we have

$$a(Ag''+Bh'')+b(Ag'+Bh')+c(Ag+Bh)=0.$$

$$\therefore a(Ag+Bh)''+b(Ag+Bh)'+c(Ag+Bh)=0.$$

7.



$v=\frac{dy}{dt}$.

$$m\frac{d^2y}{dt^2}=mg-kv.=mg-k\frac{dy}{dt}.$$

The equation is $\boxed{my''(t)+ky'(t)-mg=0}$

8 The characteristic equation is $\lambda^2-3\lambda+2=0$. $\lambda=1$ or 2

$\therefore f(x)=Ae^x+Be^{2x}$.

$f(0)=A+B=5$, $f'(0)=A+2B=-9$.

$\therefore A=19, B=-14$. $\boxed{f(x)=19e^x-14e^{2x}}$

9 $f'(x)=\frac{1}{f(x)} \iff ff'=1,\ f(x)\neq 0$.

$\therefore \frac{1}{2}(f^2(x))'=1,\ (f^2(x))'=2$.

$\therefore f^2(x)=2x+C$

$f^2(1)=2+C=3^2 \Rightarrow C=7$. $\boxed{f(x)=\sqrt{2x+7}}$ (Problem Replaced). (Check the next page for Problem #9).

$f'(1)=\frac{1}{\sqrt{2x+7}}\Big|_{x=1}=\frac{1}{3}$ satisfies the initial condition.

10 $f'(x)+P(x)f(x)=Q(x)$. $R(x)=e^{\int P(x)dx}$

$$e^{\int P(x)dx}f'(x)+e^{\int P(x)dx}P(x)f(x)=e^{\int P(x)dx}Q(x)$$

$$\left(e^{\int P(x)dx}f(x)\right)'=e^{\int P(x)dx}Q(x)$$

$$\therefore e^{\int P(x)dx}f(x)=\int e^{\int P(x)dx}Q(x)+C$$

$$\therefore f(x)=\frac{1}{R(x)}\left(\int R(x)Q(x)+C\right)$$

In Problem #2, $P(x)=1$, $Q(x)=1$, so $R(x)=e^{\int 1dx}=\tilde{C}e^x$.

$$f(x)=\frac{1}{\tilde{C}e^x}\left(\int \tilde{C}e^x+C\right)=\frac{1}{\tilde{C}e^x}\left(\tilde{C}e^x+C\right)=1+Ae^{-x}.\quad (A=C/\tilde{C}.)$$

Note that we can divide by $R(x)$ with ~~how~~ no concerns about dividing by zero because $R(x)=e^{\int P(x)dx}>0$ holds for ~~all x and~~ any real function $P(x)$ and x.

9. $2f''(x)+f'(x)=0$. The characteristic equation is $2\lambda^2+\lambda=0$.

$\therefore \lambda=0$ or $-\frac{1}{2}$. $f(x)=A+Be^{-\frac{1}{2}x}$.

$f(0)=A=5$. $f'(0)=\cancel{A}-\frac{1}{2}B=-\frac{3}{2}$.

$\therefore A=5 \quad B=3$.

$\therefore \boxed{f(x)=5+3e^{-\frac{1}{2}x}}$