MATH 19 PROBLEM SET 5
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1 Determine whether each of the following differential equations is linear or not linear. For each linear equation, specify whether the equation is homogeneous **and** whether the coefficients are constant. Note: some of the equations may not be linear as written but are equivalent to some linear equation after light rearrangement—in these cases, perform such manipulations and declare the equation to be linear. Feel free to ignore any division-by-zero concerns.

(a) $-f''''(x) + f''(x) + 7f'(x) = x^2$

(b) $f'(x) + x^2 f(x) = 0$

(c) $f''(x) = 2f'(x) - f(x)$

(d) $f(x)f'''(x) + (f(x))^2 = 0$

(e) $\dfrac{f''(x)}{f'(x)} = 7$

(f) $f'(x)f(x) + f'''(x)^3 = \frac{7}{x+1}$

2 Find a function $f(x)$ which satisfies $f'(x) + f(x) = 1$ using trial and error. (Hint: just try some simple functions you know till you find one that works. Even if you are lucky enough to guess a function that works on the first try, you should show a few functions that don't work, to illustrate the trial and error process.)

3 Find the general solution to each of the following differential equations. For each problem, show all the steps beginning with the substitution $f(x) = e^{\lambda x}$.

(a) $f''(x) + 4f'(x) + 3f(x) = 0$

(b) $f''(x) - 6f'(x) + 13f(x) = 0$

(c) $f''(x) + f(x) = 0$

(d) $f''''(x) = f(x)$

4 In Example 9.3 in the notes, we included the assumption $4km - d^2 > 0$ when solving the equation $my''(t) + dy'(t) + ky(t) = 0$, and we saw the resulting solution exhibits oscillatory behavior (meaning that it goes up and down—more precisely, that it has many local maxima and minima). Fix $m = 1$, $d = 3$, and $k = 2$ and solve the resulting differential equation to investigate what happens when $4km - d^2 < 0$. Is the resulting solution still oscillatory? Does this make sense, given the physical meaning of d in the context of Example 9.3?

5 Find a second-order linear homogeneous differential equation satisfied by $f(x) = e^x \cos(2x)$.

6 Show that if $g(x)$ and $h(x)$ are solutions to a second-order, linear homogeneous differential equation with constant coefficients, then $Ag(x) + B\,h(x)$ are also solutions, for any constants A and B. (Hint: represent the equation as $af''(x) + bf'(x) + cf(x) = 0$ and show that $Ag(x) + B\,h(x)$ satisfies it.)

7 Find the differential equation governing the motion of an object in freefall, under the influence of a constant gravitational force as well as an air resistance force proportional to the velocity of the object. You can denote the position of the object at time t as $y(t)$, the mass of the object as m, the acceleration due to gravity as g, and the constant of proportionality for the air resistance as k. (Hint: see Examples 9.2 and 9.3 in the course notes.)

8 Find a function f which satisfies the following three equations:

$$\begin{aligned} f''(x) - 3f'(x) + 2f(x) &= 0 \\ f(0) &= 5 \\ f'(0) &= -9. \end{aligned}$$

9 Find a function f satisfying

$$\begin{aligned} 2f''(x) + f'(x) &= 0 \\ f(0) &= 8 \\ f'(0) &= -3/2. \end{aligned}$$

10 In this problem, we will learn how to solve an arbitrary first-order linear differential equation:

$$f'(x) + P(x)f(x) = Q(x). \tag{1}$$

We define $R(x) = e^{\int P(x)\,dx}$ and multiply both sides of (1) by $R(x)$. Show that the left-hand side of the resulting equation is equal to $(f(x)R(x))'$. After substituting $(f(x)R(x))'$ for the left-hand side, integrate both sides and solve for $f(x)$ to show that

$$f(x) = \frac{1}{R(x)}\left(\int Q(x)R(x)\,dx + C\right)$$

solves (1). Use this formula to find all solutions to the differential equation in #2 on this problem set. Ungraded: why can we divide by $R(x)$ with no concerns about dividing by zero?