Math 19 Problem Set 4 Solutions

1\. a) $i^3 - 2 + 3i/4\,(1-11i)$

$= -i - 2 + 3i/4 - 33i^2/4$

$= -i + 3i/4 - 2 + 33/4$

$= -i/4 + 25/4$

b) $2-i\,/\,4+3i$

$= \dfrac{2-i}{4+3i}\,\dfrac{(4-3i)}{(4-3i)}$

$= \dfrac{8-10i+3i^2}{16-9i^2}$

$= 1/5 - 2/5\,i$

c) $1/a+bi$

$= \dfrac{1}{a+bi}\,\dfrac{(a-bi)}{(a-bi)}$

$= \dfrac{a-bi}{a^2-b^2i^2}$

$= a-bi\,/\,a^2+b^2$

$= \dfrac{a}{a^2+b^2} - \dfrac{b}{a^2+b^2}\,i$

d) $1+i+i^2+\cdots+i^{1000}$

note the pattern:

$i=i,\ i^2=-1,\ i^3=-i,\ i^4=1$

$\Rightarrow i+i^2+i^3+i^4=0$

So, $1+\cancel{i+i^2+i^3+i^4+\cdots+i^{1000}}$

$=1$

2\. $z = 3-2i,\quad w = 4-i$

$|z| = \sqrt{3^2+(-2)^2}$

$= \sqrt{13}$

$|w| = \sqrt{4^2+(-1)^2}$

$= \sqrt{17}$

$|z||w| = \sqrt{13}\sqrt{17}$

$= \sqrt{221}$

$\updownarrow$

$zw = (3-2i)(4-i)$

$= 12-3i-8i+2i^2$

$= 12-11i-2$

$= 10-11i$

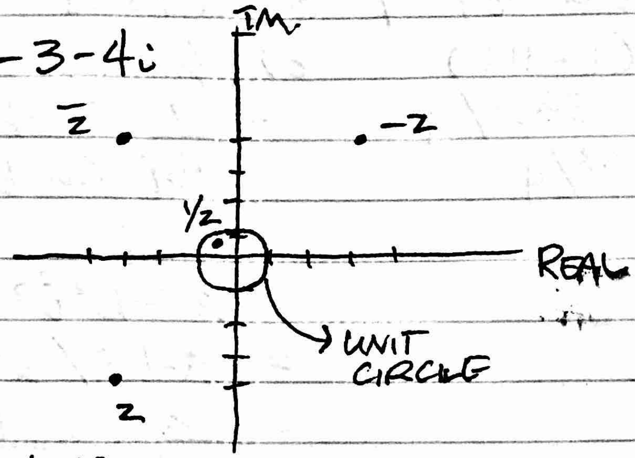
$|z||w| = |zw|$?

$|zw| = \sqrt{10^2+(-11)^2}$

$= \sqrt{221}$

✓

3. $z = -3-4i$



$z\bar{z} = |z|^2$

$\frac{1}{z} = \frac{\bar{z}}{|z|^2} = \frac{-3}{25} + \frac{4i}{25}$

4. $z = (4+2i)/5$, $w = 1 + i/2$

a) $w \Rightarrow$ left, $z \Rightarrow$ right

b) For $z = a+bi$

if $a^2+b^2 > 1$, graph will spiral outwards

if $a^2+b^2 < 1$, graph will spiral inwards

*what about $a^2+b^2 = 1$?

5. a) $z^3 = 8$

$\Rightarrow r^3 cis(3\theta) = 8 cis 0$

$r = 2$

$\theta = 0°, 120°, 240°$

solutions

$2 cis(0°) = 2$

$2 cis(120°) = -1+\sqrt{3}i$

$2 cis(240°) = -1-\sqrt{3}i$

b) $z^2 = i$

$r^2 cis(2\theta) = 1 cis(\pi/2)$

$r = 1$

$\theta = 45°, 225°$

solutions

$cis(\pi/4) = \sqrt{2}/2 + i\sqrt{2}/2$

$cis(5\pi/4) = -\sqrt{2}/2 - i\sqrt{2}/2$

5. CONT.

c) $z^8 = 1$

$r^8 \operatorname{cis}(8\theta) = 1 \operatorname{cis} 0$

$r = 1$

$\theta = 0°, 45°, 90°, 135°, 180°,$
$225°, 270°, 315°$

SOLUTIONS

$\operatorname{cis}(0°) = 1$

$\operatorname{cis}(45°) = \sqrt{2}/2 + i\sqrt{2}/2$

$\operatorname{cis}(90°) = i$

$\operatorname{cis}(135°) = -\sqrt{2}/2 + i\sqrt{2}/2$

$\operatorname{cis}(180°) = -1$

$\operatorname{cis}(225°) = -\sqrt{2}/2 - i\sqrt{2}/2$

$\operatorname{cis}(270°) = -i$

$\operatorname{cis}(315°) = \sqrt{2}/2 - i\sqrt{2}/2$

d) $z^4 + 1/z^4 = 2$

$\Rightarrow z^8 + 1 = 2z^4$

$z^8 - 2z^4 + 1 = 0$

$(z^4 - 1)^2 = 0$

so, $z^4 = 1$

$r^4 \operatorname{cis}(4\theta) = 1 \operatorname{cis} 0$

$r = 1$

$\theta = 0°, 90°, 180°, 270°$

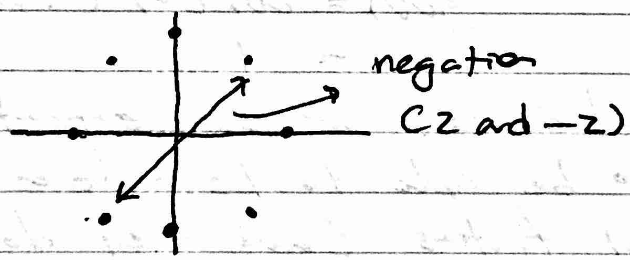
$\operatorname{cis}(0°) = 1$

$\operatorname{cis}(90°) = i$

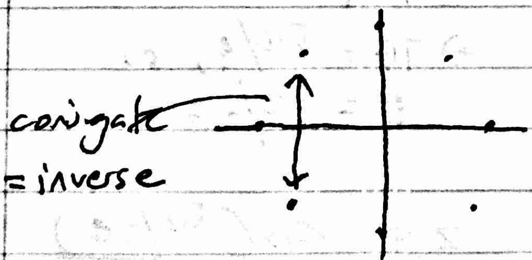
$\operatorname{cis}(180°) = -1$

$\operatorname{cis}(270°) = -i$

6.



so ~~product~~ sum = 0 because all pairs ~~are~~ on opposite ends are negations.



For a unit circle, $\bar{z} = 1/z$ (can you prove why?)
so, product $= (-1)^{n-1}$. For $n=26$, this is -1.

7. $z = r\,\text{cis}\,\theta$

in class (notes), we proved

$z^n = r^n \text{cis}(n\theta)$.

so, $z^3 = r^3 \text{cis}(3\theta)$

prove: $z^3 * (r^{-3} \text{cis}(-3\theta)) = 1$

$$\Rightarrow \frac{r^3 \text{cis}(3\theta)}{r^3 \text{cis}(3\theta)} = 1$$

✓

8. $f(\theta) = e^{k\theta}$

$f'(\theta) = d/d\theta\,(e^{k\theta})$

$= ke^{k\theta}$

$= kf(\theta)$

$f(0) = e^{k(0)}$

$= 1$

$g(\theta) = \text{cis}\,\theta$

$g'(\theta) = d/d\theta\,(\cos\theta + i\sin\theta)$

$= -\sin\theta + i\cos\theta$

$ig(\theta) = i\cos\theta - \sin\theta$

$\therefore g'(\theta) = ig(\theta)$

$g(0) = \cos(0) + i\sin(0)$

$= 1$

Given the similarities, we can expect cisθ function to be able to be represented using exponential functions (and they do!).

9. a) FALSE

$z = 3+4i,\ w = 6-4i$

$z + w = (3+4i) + (6-4i)$

$= 9$

b) FALSE

$z = (a+bi)$

$z^2 = (a+bi)(a+bi)$

$= a^2 + 2abi - b^2$

c) TRUE

$r^7 \text{cis}(7\theta) = -\sqrt{2}/2 + \sqrt{2}/2\, i$

$r = 1$

$\Rightarrow 7\theta = 5\pi/4$, so

$\theta = 5\pi/28$

$z = \text{cis}(5\pi/28)$

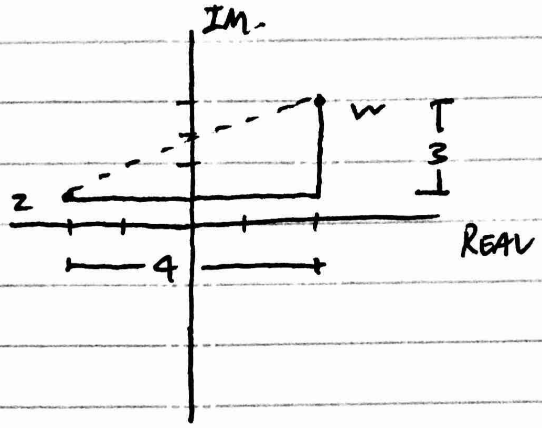
$z^2 = \sqrt{\cos^2(5\pi/28) + \sin^2(5\pi/28)}$

$= 1$

✓

10. a) $z = -2+i$

$w = 2+4i$



$|z-w|$

$= ?$ * should be 5, given picture above (3, 4, 5 right Δ)

$z-w = (-2+i)-(2+4i)$

$= -4-3i$

$|z-w| = \sqrt{16+9}$

$= \sqrt{25}$

$= 5$ ✓

b) you should draw set of all points that ~~are~~ are lying away from i and 1 at the same distance.