MATH 19 PROBLEM SET 4
FALL 2016
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1 Simplify each of the following expressions.

(a) $i^3 - 2 + \frac{3i}{4}(1 - 11i)$

(b) $\dfrac{2-i}{4+3i}$

(c) $\dfrac{1}{a+bi}$

(d) $1 + i + i^2 + i^3 + \cdots + i^{1000}$

For (c), express your answer in the form $c + di$ where c and d are both expressions involving a and b.

2 Let $z = 3 - 2i$ and $w = 4 - i$. Calculate $|z|$, $|w|$, and $|zw|$. What do you notice about the relationship between $|z||w|$ and $|zw|$?

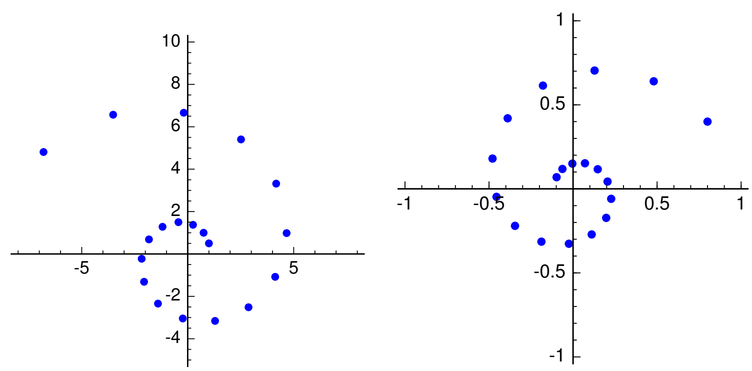
3 (a) Let $z = -3 - 4i$, and sketch the point z, its negation $-z$, its conjugate $\overline{z} = -3 + 4i$, and its inverse $\frac{1}{z}$ in the complex plane.

(b) Show that z and $\frac{1}{z}$ cannot both be inside the unit circle, if z is a complex number.

(c) Show that z and $\frac{1}{z}$ cannot both be outside the unit circle, if z is a complex number.

(d) Show that z and $\frac{1}{z}$ cannot be on the same side of the real axis or on opposite sides of the imaginary axis, if z is a complex number.

4 We set $z = \frac{4+2i}{5}$ and $w = 1 + \frac{i}{2}$. The numbers $z, z^2, z^3, \ldots, z^{20}$ are plotted in one of the two figures below, and the numbers $w, w^2, w^3, \ldots, w^{20}$ are plotted in the other.



(a) Which figure corresponds to z and which corresponds to w? (b) How can you tell for an arbitrary complex number z whether the plot of $z, z^2, z^3, \ldots,$ will spiral outward towards infinity or spiral inward towards zero? (Note: the goal here is to come up with a statement of the form "if z satisfies [insert simple condition to check], then the numbers $z, z^2, z^3, \ldots,$ spiral outward, and if z satisfies [insert another simple condition], then the numbers $z, z^2, z^3, \ldots,$ spiral inward.")

5 Solve each of the following equations. In each case, you should express all solutions in Cartesian form $a + bi$, where a and b are real numbers.

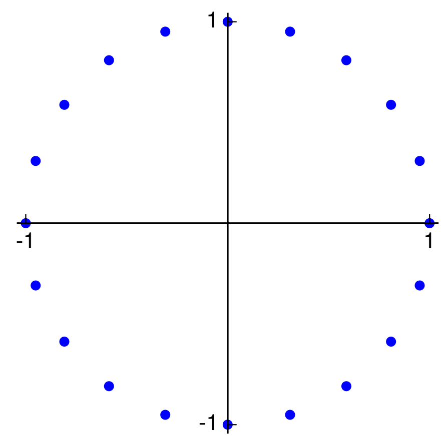
(a) $z^3 = 8$

(b) $z^2 = i$

(c) $z^8 = 1$

(d) $z^4 + \frac{1}{z^4} = 2$

6 The 20th roots of unity are shown in the figure below



What is the sum of these 20 numbers (hint: consider 3(a))? What is the product of these 20 numbers (hint: consider a different part of 3(a))? Hint for both: find good ways to sum/multiply the numbers in *pairs*.

7 In this problem, we will see that negative exponents (as well as positive ones) pass to the argument of cis: show that if $z = r \operatorname{cis} \theta$, then $z^{-3} = r^{-3} \operatorname{cis}(-3\theta)$. (Hint: show that z^3 times $r^{-3} \operatorname{cis}(-3\theta)$ is equal to 1).

8 Let k be a fixed number, and show that $f(\theta) = e^{k\theta}$ satisfies the equations $f'(\theta) = kf(\theta)$ and $f(0) = 1$. Show that $g(\theta) = \operatorname{cis} \theta$ satisfies[1] the equations $g'(\theta) = ig(\theta)$ and $g(0) = 1$. Ungraded: what does this suggest about how $\operatorname{cis} \theta$ might be representable using e?

9 *True or false*. For each of the following statements, either show that it is true, or else demonstrate that it is false by providing a counterexample.

(a) If z and w are complex numbers and $z + w$ is real, then z and w are complex conjugates.

(b) If z is a nonzero complex number, then z^2 is a positive real number

(c) If $z^7 = -\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$, then $|z| = 1$.

10 The distance between two points z and w in the complex plane is given by $|z - w|$. (a) Verify this statement in the case where $z = -2 + i$ and $w = 2 + 4i$. (b) Use this statement to draw a picture of the set of all points in the plane satisfying the equation $|z - i| = |z - 1|$.

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[1] We haven't discussed the derivative of a complex-valued function, but it's defined exactly how you would expect: you differentiate the real and imaginary parts separately. So, for example, the derivative of $x^3 + x^2 i$ with respect to x would be $3x^2 + 2xi$.