MATH 0190 Homework Assignment 3.
Solutions

1. $W = \int_6^{100} F\,dx = \int_6^{100} 300\frac{6m}{x}\,dx = 1800m \ln\frac{100}{6} = \boxed{1800\ln\frac{50}{3}\ (\text{joules})}$

2. (a). The potential energy P is $\frac{1}{2}kx^2$.
Since $kx=60$, $P = \frac{1}{2}kx^2 = \frac{60}{2x}\cdot x^2 = 30x.$ (Joules).
(b). The change in gravitational potential energy is
$P' = \boxed{60x\cdot g\ (\text{joules})}$
(c) (optional). There is actually some energy dissipation in the spring.

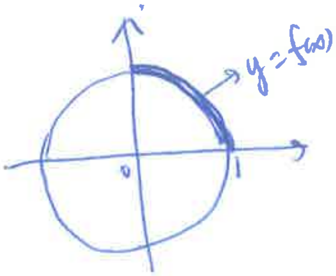
3. $W = \int F\,dx = \int_0^4 g\cdot 1000\cdot 800\cdot(5-x)\,dx = \boxed{g\times 10^5\times 9.6\ (\text{Joules})}$
$= \boxed{960000\,g\ (\text{Joules})}$

(4). The length of the graph of $f(x)=\frac{1}{8}x^2-\ln x$ over $[1,2]$ is

$$\int_1^2\sqrt{f'(x)^2+1}\,dx=\int_1^2\sqrt{(\tfrac{1}{4}x-\tfrac{1}{x})^2+1}\,dx=\int_1^2\sqrt{(\tfrac{1}{4}x+\tfrac{1}{x})^2}\,dx$$

$$=\int_1^2(\tfrac{1}{4}x+\tfrac{1}{x})\,dx=(\tfrac{1}{4}\cdot\tfrac{1}{2}x^2+\ln x)\Big|_1^2=\boxed{\tfrac{3}{8}+\ln 2}$$

(5) It suffices to show that the length of the following graph of $f(x)=\sqrt{1-x^2}$ over the interval $[0,1)$ is $\pi/2$. (Denote this length to be ℓ).



Indeed, $f'(x)=\dfrac{-x}{\sqrt{1-x^2}}$. So we have

$$\ell=\int_0^1\sqrt{\frac{x^2}{1-x^2}+1}\,dx=\int_0^1\frac{1}{\sqrt{1-x^2}}\,dx.$$

Let $x=\sin\theta$, then $\ell=\int_0^{\pi/2}\frac{1}{\cos\theta}\cos\theta\,d\theta=\pi/2$.

(6) In Cartesian coordinate system, the equation of the graph is $x=3$. Since we have $x=r\cos\theta$, $y=r\sin\theta$, $r\cos\theta=3$ is equivalent to $x=3$.

$\therefore$ $r\cos\theta=3$ is the equation that express the graph. $r=\frac{3}{\cos\theta}$.

$\therefore$ $\boxed{F(\theta)=3/\cos\theta}$ is the desired function.

(7) Use $x=r\cos\theta$, $y=r\sin\theta$, $r=\sqrt{x^2+y^2}$, $\tan\theta=y/x$ (if $x\neq 0$).

(a). $(x,y)=\boxed{(0,2)}$ (b) $(x,y)=\boxed{(\sqrt{10},\sqrt{10})}$

(c) $(x,y)=\boxed{(-3\pi,0)}$ (d). $(r,\theta)=\boxed{(2,\frac{7\pi}{6})}$

(e) $(r,\theta)=\boxed{(4,\pi)}$ (f) $(r,\theta)=\boxed{(5,\arctan\frac{4}{3})}$

(8) $0<e<1$. $r=\dfrac{1}{1-e\cos\theta}$. $\cos\theta=\dfrac{x}{r}$.

$r=\dfrac{1}{1-e\cos\theta}\Rightarrow r-re\cos\theta=1\Rightarrow r=1+er\cos\theta=1+ex\Rightarrow r^2=(1+ex)^2.$

$\Rightarrow x^2+y^2=(1+ex)^2\Rightarrow(1-e^2)x^2-2ex+y^2=1\Rightarrow(1-e^2)(x-\frac{e}{1-e^2})^2+y^2=1+\frac{e^2}{1-e^2}$

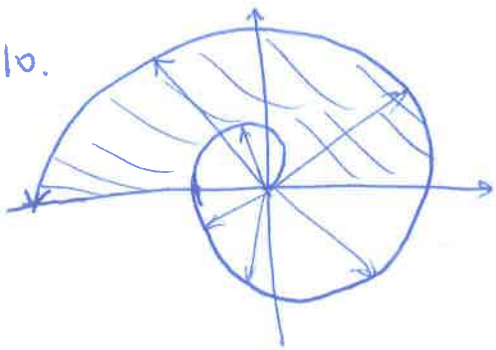
$$\Rightarrow\frac{(x-\frac{e}{1-e^2})^2}{(1+\frac{e^2}{1-e^2})/(1-e^2)}+\frac{y^2}{1+\frac{e^2}{1-e^2}}=1.$$

9. The area of the shaded region is (Use Th. 7.1 in the note).

$$\int_0^{2\pi/10} \frac{1}{2}\sin^2(5\theta)\,d\theta$$

$$= \frac{1}{2}\int_0^{2\pi/10} \frac{1-\cos(10\theta)}{2}\,d\theta.$$

$$= \boxed{\frac{\pi}{20}.}$$

10. 

The area of the dark region is

$$\int_{2\pi}^{3\pi} \frac{1}{2}\theta^2\,d\theta - \int_0^{\pi} \frac{1}{2}\theta^2\,d\theta.$$

$$= \frac{1}{6}\theta^3\Big|_{2\pi}^{3\pi} - \frac{1}{6}\theta^3\Big|_0^{\pi}. = \boxed{3\pi^3.}$$