Solutions to MATH 19 Problem Set 1.

Zhiyuan Zhang

1. (Review) Evaluate the following integrals:

(a) $\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$.

- Let $\sqrt{x}=t$, then $\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx = \int \frac{\sin t}{t}\,d(t^2) = \int \frac{\sin t}{t}\cdot 2t\,dt = 2\int \sin t\,dt$.

which gives $\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx = -2\cos t + C = \boxed{-2\cos\sqrt{x}+C}$.

(b) $\int \frac{1}{1-e^{-x}}\,dx$

- Let $1-e^{-x}=t$, then $dt = -e^{-x}\cdot(-1)\,dx = e^{-x}dx$, $dx = e^{x}dt$, and we ~~have~~ have $e^{x} = \frac{1}{1-t}$.

$$\therefore \int \frac{1}{1-e^{-x}}\,dx = \int \frac{1}{t}e^{x}dt = \int \frac{1}{t}\frac{1}{1-t}\,dt = \int\left(\frac{1}{t}+\frac{1}{1-t}\right)dt = \ln|t| - \ln|1-t| + C$$

$$= \ln|1-e^{-x}| - \ln e^{-x} + C$$

$$= \ln|1-e^{-x}| + x + C$$

$$\boxed{= \ln|1-e^{x}| + C}$$

(c) $\int_{-1}^{3} e^{|x|}\,dx$.

- Since $|x| = \begin{cases} x. & x\geq 0. \\ -x. & x<0. \end{cases}$ we can write

$$\int_{-1}^{3} e^{|x|}\,dx = \int_{0}^{3} e^{|x|}\,dx + \int_{-1}^{0} e^{|x|}\,dx = \int_{0}^{3} e^{x}\,dx + \int_{-1}^{0} e^{-x}\,dx.$$

$$= e^{x}\Big|_{0}^{3} + (-e^{-x})\Big|_{-1}^{0} = e^{3}-e^{0} + (-e^{0}) - (-e^{1}) = \boxed{e^{3}+e-2}.$$

(d) $\int_{0}^{1/2} \frac{1}{1-x^2}\,dx$

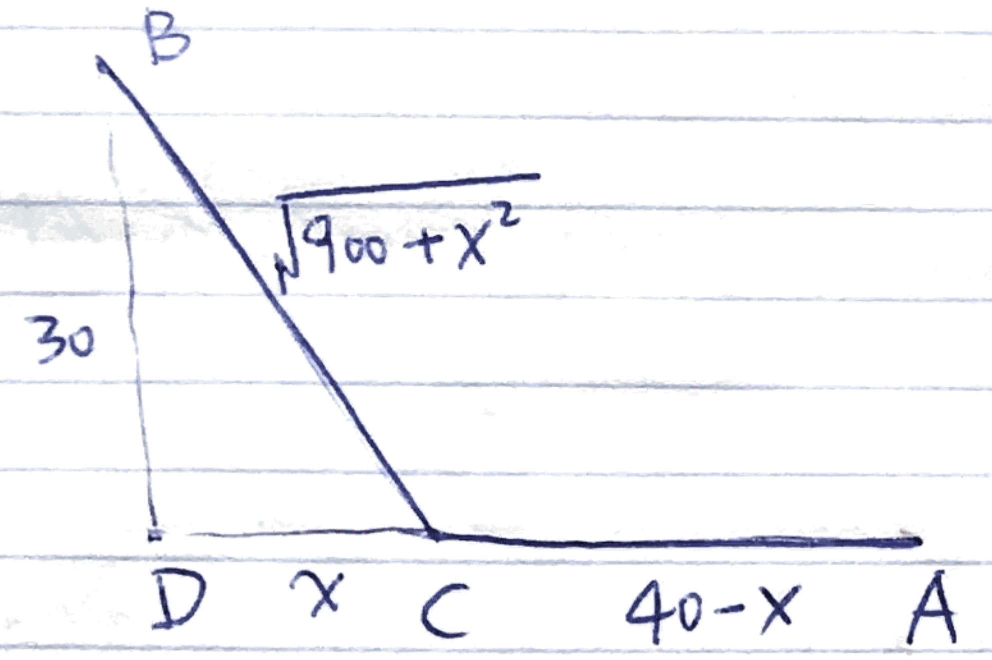
- Note that $\frac{1}{1-x^2} = \frac{1}{1-x}\cdot\frac{1}{1+x} = \frac{1}{2}\left(\frac{1}{1-x}+\frac{1}{1+x}\right)$, so we have

$$\int_{0}^{1/2} \frac{1}{1-x^2}\,dx = \frac{1}{2}\int_{0}^{1/2}\left(\frac{1}{1-x}+\frac{1}{1+x}\right)dx$$

$$= \frac{1}{2}\left(-\log(1-x)\Big|_{x=0}^{x=1/2} + \log(1+x)\Big|_{x=0}^{x=1/2}\right).$$

$$= \frac{1}{2}\left(\log 2 + \log\frac{3}{2}\right) = \boxed{\frac{1}{2}\log 3}.$$

2.



The dog wants to run to C and then swim from C to B, where he can get the ball.

(a) Since AD is 40 meters, it's easy to see that AC is $(40-x)$ meters. Then the time for the dog to run from A to C is

$$\boxed{t_{AC} = \frac{40-x}{6} \text{ (seconds.)}}$$

(b) By Pythagorean Theorem, the distance from C to B is $\sqrt{30^2+x^2}$ meters.
So the time for the dog to ~~sw~~ swim from C to B is

$$\boxed{t_{CB} = \frac{\sqrt{30^2+x^2}}{3} \text{ (seconds).}}$$

(c). ~~Let~~ Let T be the total travel time, then we have

$$T = t_{AC} + t_{CB}.$$

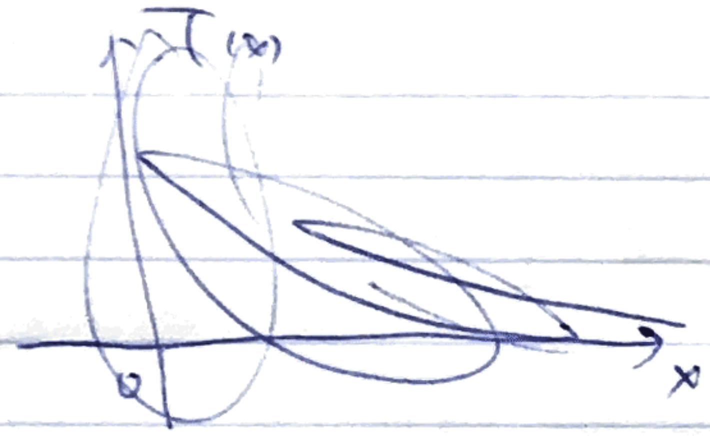
Therefore, $T = T(x) = \dfrac{40-x}{6} + \dfrac{\sqrt{30^2+x^2}}{3}$. We want to minimize T.

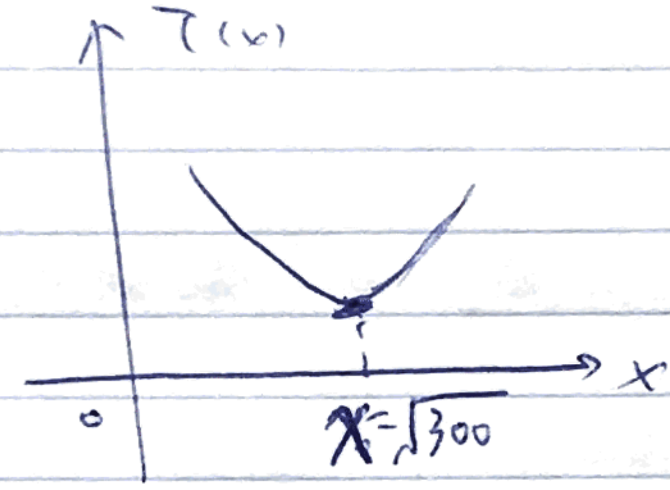
Take the derivative with respect to x, we find

$$T'(x) = \frac{dT}{dx}(x) = -\frac{1}{6} + \frac{1}{3}\cdot\frac{1}{\sqrt{900+x^2}}\cdot x.$$

Let $T'(x)=0$, we can ~~see that~~ solve out that $x = \sqrt{300}$. (meters).

By checking the second derivative ($T''(x) = \frac{1}{3}\left(\sqrt{900+x^2} - \frac{x^2}{\sqrt{900+x^2}}\right)\frac{1}{900+x^2} > 0.$), ~~we~~ we can find that the graph of $T(x)$ ~~is~~ looks like:





So $\boxed{x = \sqrt{300} = 10\sqrt{3} \text{ (meters)}}$, is the value that minimizes $T(x)$.

③ Evaluate the integrals:

(a) $\int x^2 e^x \, dx$.

· Note that $e^x dx = d(e^x)$, so $\int x^2 e^x dx = \int x^2 d(e^x) = x^2 e^x - \int e^x d(x^2)$.

$\therefore \int x^2 e^x dx = x^2 e^x - 2\int x e^x dx + C_1$

Do the integration by parts again, we have

$$\int x^2 e^x dx = x^2 e^x - 2\int x \, d(e^x) = x^2 e^x - 2\left(x e^x - \int e^x dx\right) + C_2.$$

$$\therefore \int x^2 e^x dx = \boxed{x^2 e^x - 2x e^x + 2e^x + C}.$$

(b) $\int_0^3 \ln(x^2+1)\, dx$.

· Integrate by parts. We obtain:

$$\int_0^3 \ln(x^2+1)\, dx = x\ln(x^2+1)\Big|_0^3 - \int_0^3 x \, d\ln(x^2+1).$$

$$= x\ln(x^2+1)\Big|_0^3 - \int_0^3 x \cdot \frac{2x}{x^2+1}\, dx.$$

$$= x\ln(x^2+1)\Big|_0^3 - 2\int_0^3 \frac{x^2}{x^2+1}\, dx.$$

Note that $x^2 = (x^2+1) - 1$.

$$\therefore \int_0^3 \ln(x^2+1)\, dx = x\ln(x^2+1)\Big|_0^3 - 2\int_0^3 \frac{(x^2+1)-1}{x^2+1}\, dx.$$

$$= x\ln(x^2+1)\Big|_0^3 - 2\int_0^3 1 \cdot dx + 2\int_0^3 \frac{1}{x^2+1}\, dx.$$

$$= x\ln(x^2+1)\Big|_0^3 - 2\cdot 3 + 2\arctan x\Big|_0^3.$$

$$= \boxed{3\ln 10 + 2\arctan 3 - 6.}$$

(c) $\int_0^1 x\arctan x\,dx.$

Integrating by parts, we have

$$\int_0^1 x\arctan x\,dx = \int_0^1 \arctan x\, d\left(\tfrac{1}{2}x^2\right).$$

$$= \arctan x\cdot\tfrac{1}{2}x^2\Big|_0^1 - \int_0^1 \tfrac{1}{2}x^2\, d(\arctan x).$$

$$= \arctan x\cdot\tfrac{1}{2}x^2\Big|_0^1 - \int_0^1 \tfrac{1}{2}x^2\frac{1}{1+x^2}\,dx.$$

$$= \arctan x\cdot\tfrac{1}{2}x^2\Big|_0^1 - \tfrac{1}{2}\int_0^1\frac{(x^2+1)-1}{1+x^2}\,dx.$$

$$= \arctan x\cdot\tfrac{1}{2}x^2\Big|_0^1 - \tfrac{1}{2}\int_0^1 1\cdot dx + \tfrac{1}{2}\int_0^1\frac{1}{1+x^2}\,dx.$$

$$= \arctan x\cdot\tfrac{1}{2}x^2\Big|_0^1 - \tfrac{1}{2}\cdot 1 + \tfrac{1}{2}\arctan x\Big|_0^1.$$

$$= \frac{\pi}{4}\cdot\frac{1}{2} - \frac{1}{2} + \frac{1}{2}\cdot\frac{\pi}{4}.$$

$$= \boxed{\frac{\pi}{4} - \frac{1}{2}.}$$

(d). $\int x\sqrt{x+3}\,dx.$

Let $\sqrt{x+3} = t$, then $x = t^2-3$, $dx = 2t\,dt$.

$$\therefore \int x\sqrt{x+3}\,dx = \int (t^2-3)\,t\cdot 2t\,dt.$$

$$= 2\int t^4\,dt - 6\int t^2\,dt.$$

$$= \frac{2}{5}t^5 - \frac{6}{3}t^3 + C.$$

$$= \boxed{\frac{2}{5}(x+3)^{5/2} - 2(x+3)^{3/2} + C.}$$

(e) $\int \sin(\ln x)\,dx$

Let $\ln x = t$, then $x = e^t$, $dx = e^t\,dt$.

$\cancel{=\int \sin t}$ $\qquad = \int \sin t\, e^t\, dt$

$\therefore \int \sin(\ln x)\,dx = \int \sin t\, de^t = e^t \sin t - \int e^t\, d\sin t = e^t \sin t - \int \cos t\, de^t + C_1$

$= e^t \sin t - e^t \cos t + \int e^t\, d\cos t = C_2 + e^t \sin t - e^t \cos t - \int e^t \sin t\, dt.$

Hence we have $\int \sin t\, \cancel{de^t}\, e^t\, dt = C_2 + e^t(\sin t - \cos t) - \int \sin t\, e^t\, dt$ ①, and

$\int \sin(\ln x)\,dx = \int \sin t\, e^t\, dt$ ②.

From ①, we obtain $\int \sin t\, e^t\, dt = \frac{1}{2} e^t(\sin t - \cos t) + C$.

$\therefore \int \sin(\ln x)\,dx = \int \sin t\, e^t\, dt = \frac{1}{2} e^t(\sin t - \cos t) + C = \boxed{\frac{1}{2} x(\sin \ln x - \cos \ln x) + C}$

(f) $\int (\ln x)^2 dx$.

- Let $\ln x = t$, then $x = e^t$, $dx = e^t dt$.

$\therefore \int (\ln x)^2 dx = \int t^2 e^t\, dt = \int t^2 de^t = t^2 e^t - \int e^t\, d(t^2) + C_1.$

$= t^2 e^t - 2\int t e^t\, dt + C_1.$

$= t^2 e^t - 2\int t\, de^t + C_1.$

$= t^2 e^t - 2te^t + 2\int e^t dt + C_2.$

$= t^2 e^t - 2te^t + 2e^t + C_2.$

$\therefore \int (\ln x)^2 dx = \boxed{(\ln x)^2 x - 2\ln x \cdot x + 2x + C}$

(g) $\int_0^\pi e^x \cos x\, dx$.

- Integrating by parts, we have

$\int_0^\pi e^x \cos x\, dx = \int_0^\pi \cos x\, de^x = \cos x \cdot e^x \Big|_0^\pi - \int_0^\pi \sin x\, e^x dx \cdot (-1)$. $\quad -\int_0^\pi e^x \cos x\, dx$

$= \cos x \cdot e^x \Big|_0^\pi + \int_0^\pi \sin x\, de^x = \cos x \cdot e^x \Big|_0^\pi + \sin x\, e^x \Big|_0^\pi - \cancel{\int_0^\pi}$

Therefore $\int_0^\pi e^x \cos x\, dx = \frac{1}{2} \cos x \cdot e^x \Big|_0^\pi + \sin x \cdot e^x \Big|_0^\pi = \boxed{-(e^\pi + 1)/2}.$

4. ① If we integrate by parts, then

$$\int e^x \sin x\,dx = \int \sin x\,de^x = e^x \sin x - \int e^x \cos x\,dx$$

$$= e^x \sin x - \int \cos x\,de^x = e^x \sin x - e^x \cos x - \int e^x \sin x\,dx + C_1.$$

$$\therefore \int e^x \sin x\,dx = \frac{1}{2} e^x (\sin x - \cos x) + C.$$

② On the other hand, we can assume there is an anti-derivative of the form $f(x) = Ae^x \sin x + Be^x \cos x$ for $e^x \sin x$, i.e. $f'(x) = e^x \sin x$.

$$\therefore f'(x) = Ae^x \sin x + Ae^x \cos x + Be^x \cos x + Be^x(-\sin x)$$

$$= e^x \sin x \cdot (A-B) + e^x \cos x \cdot (A+B).$$

$$= e^x \sin x.$$

This holds for all x, so there must be $A-B=1$, $A+B=0$, which gives $A = 1/2$, $B = -1/2$.

$$\therefore \int e^x \sin x\,dx = f(x) + C = \frac{1}{2} e^x (\sin x - \cos x) + C.$$

These two methods gives the same answer, but integration by part is easier.

5 We wait to prove that $\int x^n \cos x\,dx = x^n \sin x - n\int x^{n-1}\sin x\,dx$.

For this we do integration by parts:

$$\int x^n \cos x\,dx = \int x^n \, d\sin x = x^n \sin x - \int \sin x\, d(x^n) = x^n \sin x - n\int x^{n-1} \sin x\, dx.$$

6. Integrate by parts, we have:

$$\int f(x) g'(x)\,dx = \int f(x)\, dg(x) = \cancel{f(x)g(x)}$$

$$= f(x) g(x) - \int g(x)\, df(x).$$

$$= f(x) g(x) - \int f'(x) g(x)\, dx.$$

$$= f(x) g(x) - \int f'(x)\, dG(x).$$

$$= f(x) g(x) - G(x) f'(x) + \int G(x) f''(x)\, dx.$$

which gives the desired equality.