MATH 19 PROBLEM SET 11
FALL 2016
BROWN UNIVERSITY
SAMUEL S. WATSON

1 Show that if f is an odd 2π-periodic function, then $a_n = 0$ for all $n \geq 0$. Show that if f is an even 2π-periodic function, then $b_n = 0$ for all $n \geq 1$. Here a_n and b_n denote the Fourier coefficients of f, as in Definition 13.3 in the notes.

2 Verify using the definition of (real) Fourier coefficients that the second-order approximation of $f(x) = 5 + 2\sin x + 3\cos 2x$ is equal to f. (Feel free to use Theorem 13.1.)

3 Consider a 2π-periodic function f such that $f(x) = x$ for all $0 \leq x < \pi$ and $f(x) = 0$ for all $\pi < x < 2\pi$.

(a) Find the Fourier series of f.

(b) Suppose that the Fourier series for f, evaluated at x, converges to $f(x)$ for all $x \in \mathbb{R}$. Find $f(\pi)$.

4 Consider the squave-wave function f whose real Fourier series is calculated on page 69 in the course notes. Calculate the complex Fourier coefficients of f and verify that the relation between c_n and (a_n, b_n) in Theorem 13.5 is satisfied.

5 Find the complex Fourier coefficients of the 2π-periodic function which is equal to 0 on $[-\pi, 0)$ and $x(1-x)$ on $[0, \pi]$. (Note: it gets a little messy. Towards the end, you should simplify each term, but don't bother collecting terms.)

6 Convert each of the following trigonometric polynomials to exponential form, using the identities $\sin x = \frac{e^{ix} - e^{-ix}}{2i}$ and $\cos x = \frac{e^{ix} + e^{-ix}}{2}$.

(a) $2 - \sin 3x - \sin 5x$

(b) $\frac{1}{2}\cos x - \frac{1}{2}\sin x$

7 Convert each of the following complex Fourier approximations to real form, using Euler's formula.

(a) $3e^{-ix} + 3e^{ix}$

(b) $(1+i)e^{3ix} + (1-i)e^{-3ix} + ie^{5ix} - ie^{-5ix}$

8 If $L > 0$ and f is $2L$-periodic, then its Fourier coefficients are

$$a_0 = \frac{1}{2L}\int_0^{2L} f(x)\,dx$$
$$a_n = \frac{1}{L}\int_0^{2L} f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx$$
$$b_n = \frac{1}{L}\int_0^{2L} f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx$$

Use these formulas to find the Fourier coefficients of the 1-periodic function f which is equal to x on $[0, 1)$

9 Consider a guitar string with tension T and linear mass density μ. The motion of the string when plucked is determined by the *wave equation*. Solving the wave equation tells us that if the original configuration position of the string satisfies

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx,$$

for $0 \leq x \leq \pi$, then the sound produced by the string is given by

$$s(t) = \sum_{n=1}^{\infty} b_n e^{-n\gamma t} \cos(2\pi n f_1 t), \tag{1}$$

where γ is a constant, called the *dampening factor*, and $f_1 = \frac{1}{2\pi}\sqrt{\frac{T}{\mu}}$ is the *fundamental frequency* of the string (the main note you hear when you play it), and t is the number of seconds after release of the string. Suppose the plucking displacement f is the function whose graph consists of line segments connecting $(0,0)$, $(\pi/2, \pi/2)$, and $(\pi, 0)$.

(a) Extend the domain of f to the whole real line in such a way that f is odd and 2π-periodic. Hint: draw the graph of f over $[0, \pi]$; then what does the graph have to look like over interval $[-\pi, 0]$?

(b) Calculate f's Fourier coefficients to find b_n.

(c) Assuming $\gamma = 1.7$, find the number of nonzero terms in the infinite sum (1) which have a coefficient $b_n e^{-n\gamma t}$ at least 0.01% as large as the coefficient of the fundamental frequency term, one second after plucking.

10 Consider a circuit with a 0.25 farad capacitor, a 1.0 ohm resistor, and 1.0 henry inductor. Suppose the circuit is hooked up to a voltage source $V(t)$ which applies a unit voltage for π seconds, then no voltage of π seconds, unit voltage for π seconds, no voltage for π seconds, and so on. Express the steady state solution of the differential equation governing the flow of charge $Q(t)$ in the system, as an infinite, real Fourier series.

Hint: begin by calculating the complex Fourier coefficients of V, transform them in the manner developed in the solution of Example 13.8, and translate back to a real Fourier series. Feel free to use Theorem 13.5 and Computational Investigation 5 instead of calculating the Fourier coefficients from scratch.