MATH 19 PROBLEM SET 10
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You will have to run some code to complete this problem set. You will not have to learn any programming; at most you are being asked to do some very light editing to given code. All code can be run in-browser at sagecell.sagemath.org. If there is some issue with that website, or if you prefer a more full-featured environment, check out cloud.sagemath.com. Furthermore, you do not need to include any digital output directly in your solutions; you may carefully sketch the graphs instead.

$\boxed{1}$ Find the exact values of the infinite series (a) $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{5^n n!}$ and (b) $\displaystyle\sum_{n=0}^{\infty} \frac{(n+1)(n+2)}{2^n}$. Hint for (b): start with the Taylor series for $\frac{1}{1-x}$ and calculate its second derivative term-by-term.

$\boxed{2}$ Find the least positive integer n such that the nth order Taylor polynomial $P_n(x)$ of $\sin(x^4)$ centered at 0 has the property that

$$\left| \int_0^1 \sin(x^4)\,dx - \int_0^1 P_n(x)\,dx \right| < 10^{-4}.$$

Sketch a graph of $\sin(x^4)$ and $P_n(x)$, for this value of n. Hints: the line (you can run this code at sagecell.sagemath.org)

```
n(integrate(sin(x^4),x,0,1)) # n means "convert to decimal"
```

returns an accurate decimal approximation of the specfied integral. The line

```
n(integrate(1/3*x^2 - 1,x,0,1))
```

returns the integral of the polynomial $\frac{1}{3}x^2 - 1$ (which was chosen arbitrarily, for purposes of illustration). Replace `1/3*x^2 - 1` with the nth order Taylor polynomial of $P_n(x)$ to solve this problem. Once you have found the polynomial you want, you can run

```
plot([sin(x^4), 1/3*x^2 - 1],x,0,1)
```

to see a picture (again you want to replace `1/3*x^2 - 1` with the desired polynomial).

$\boxed{3}$ Find $f^{(11)}(0)$, where $f(x) = (1+x^3)^{44}$.

$\boxed{4}$ Suppose that $f(x) = \sum_{n=0}^{\infty} a_n x^n$ is a function which satisfies the differential equation $f''(x) = -f(x)$, with $f(0) = 0$ and $f'(0) = 1$. Use the initial data to find a_0 and a_1, then substitute the power series for f into the differential equation and equate coefficients to find a_2. Repeat to find a_3, and continue until you recognize the pattern and can write down an expression for a_n. What is the name of this function f?

$\boxed{5}$ (a) Fix $z = 5 + i$ and $w = 239 - i$. Show that the angular coordinate of $z^4 w$ is equal to $\frac{\pi}{4}$. If you'd rather not do this by hand, you can run

```
(5 + i)^4 * (239 - i)
```

in Sage. Show that the angular polar coordinates of z and w are $\arctan(1/5)$ and $-\arctan(1/239)$, respectively. Use these observations to show that

$$\frac{\pi}{4} = 4\arctan\frac{1}{5} - \arctan\frac{1}{239}.$$

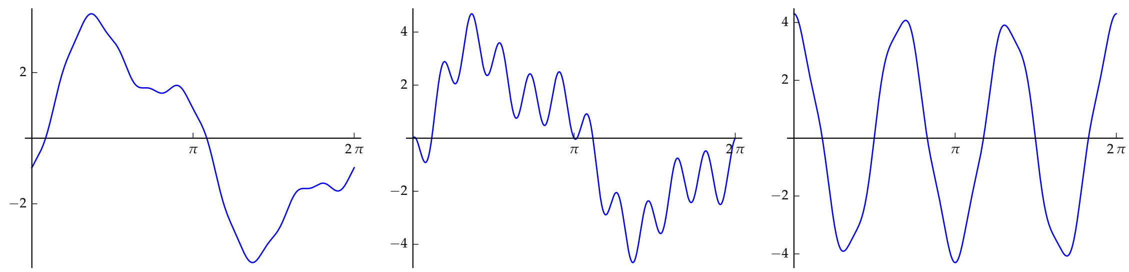
Substitute the fifth-order Maclaurin polynomial for arctan into this equation to obtain an approximation of π. What is the difference between this approximation and the actual value of π?

(b) Use the identity $\pi = 4\arctan(1)$ to obtain an approximation for π by substituting 1 into the fifth-order Maclaurin polynomial for arctan. What is the difference between this approximation and the actual value of π?

The following lines of Sage code might be helpful to you.

```
p(x) = taylor(arctan(x),x,0,5)
n( 4*p(1/5)-p(1/239) - pi/4 )
```

$\boxed{6}$ The functions $a\sin(x) + b\cos(3x) + c\cos(11x)$ are shown below, where (a, b, c) is equal to $(0.1, 4, 0.3)$, $(3, -1, 0.1)$, and $(3, -1, 1)$, not necessarily in that order.



Match each triple of coefficients to its graph. Explain your reasoning carefully. In other words, discuss features in each graph which indicate which triple it corresponds to.

$\boxed{7}$ Show that if g is a 2π-periodic function, then

$$\int_0^{2\pi} g(x)\,dx = \int_a^{a+2\pi} g(x)\,dx$$

for all $a \in \mathbb{R}$. Substitute $a = -\pi$ and $g(x) = f(x)\sin nx$ to conclude that the intervals of integration in the definitions of the Fourier coefficients $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ for a 2π-periodic function f can be changed from $[0, 2\pi]$ to $[-\pi, \pi]$ without changing the values of the coefficients. (Hint: graphing here will be key: draw the graph of a 2π-periodic function g and sketch the area under the graph over the interval $[0, 2\pi]$ and over $[a, a + 2\pi]$. Why are these areas the same? Once you've figured this out, perform an appropriate substitution to make it rigorous.)

$\boxed{8}$ Find the Fourier series of the 2π-periodic function f which is equal to 0 on the interval $[-\pi, 0)$ and $\cos x$ on $[0, \pi)$. For which $x \in \mathbb{R}$ does the Fourier series converge to $f(x)$? Sketch a plot of the function and its fourth-order Fourier approximation. The following Sage code might be helpful:

```
f = piecewise([((-pi,0),0),((0,pi),cos(x))])
plot([f,f.fourier_series_partial_sum(4,pi)],-pi,pi)
```

$\boxed{9}$ Find the Fourier series of the 2π-periodic function f which is equal to x on the interval $[-\pi, \pi)$. For which $x \in \mathbb{R}$ does the Fourier series converge to $f(x)$? Sketch a plot of the function and its fourth-order Fourier approximation. The following Sage code might be helpful:

```
f = piecewise([((-pi,pi),x)])
plot([f,f.fourier_series_partial_sum(4,pi)],-pi,pi)
```

$\boxed{10}$ In this problem, we find the function of the form

$$P(x) = a_0 + a_1\cos x + b_1\sin x$$

which mostly closely approximates a given function f on $[0, 2\pi]$ in that sense that $\int_0^{2\pi}(P(x) - f(x))^2\,dx$ is as small as possible.

(a) Show that for any functions P and f, we have

$$\int_0^{2\pi}(P(x) - f(x))^2\,dx = \int_0^{2\pi} P(x)^2\,dx - 2\int_0^{2\pi} P(x)f(x)\,dx + \int_0^{2\pi} f(x)^2\,dx. \qquad (1)$$

(b) Show that $\int_0^{2\pi} P(x)^2\,dx = 2\pi a_0^2 + \pi a_1^2 + \pi b_1^2$.

(c) Show that $\int_0^{2\pi} P(x)f(x)\,dx = a_0 \int_0^{2\pi} f(x)\,dx + a_1 \int_0^{2\pi} f(x)\cos x\,dx + b_1 \int_0^{2\pi} f(x)\sin x\,dx$.

(d) After substituting (c) and (b) into (1), the right-hand side is a quadratic function of a_0. Find the value of a_0 which minimizes this function. (This might seem intimidating because of all the notation, but remember that you know from single-variable calculus how to minimize a quadratic function.)

(e) Repeat (d) with a_1 and b_1.

(f) Draw a conclusion about the function $P(x)$ which minimizes $\int_0^{2\pi} (P(x) - f(x))^2\,dx$.