DATA 1010
Problem Set 5
Due 12 October 2018 at 11 PM

Problem 1

Suppose that X and Y are random variables whose joint distribution is given by the following table.

		X			
		-1	0	1	2
	-1	0	$\frac{1}{36}$	$\frac{1}{6}$	$\frac{1}{12}$
Y	0	$\frac{1}{18}$	0	$\frac{1}{18}$	0
	1	0	$\frac{1}{36}$	$\frac{1}{6}$	$\frac{1}{12}$
	2	$\frac{1}{12}$	0	$\frac{1}{12}$	$\frac{1}{6}$

(a) Find $P(X \geq 1 \text{ and } Y \leq 0)$.

(b) What is the conditional probability of the event $\{Y \leq 0\}$ given that $X = 2$?

(c) Are X and Y independent?

(d) What is the distribution of $Z = XY$?

Problem 2

A die is rolled twice. Let X denote the sum of the two numbers that turn up, and Y the difference of the numbers (first roll minus second). Show that $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$ but that X and Y are not independent.

Problem 3

Consider m zeros and n ones arranged in order uniformly at random. For example, if $m = 3$ and $n = 7$, then

$$1, 0, 1, 1, 0, 0, 1, 1, 1, 1$$

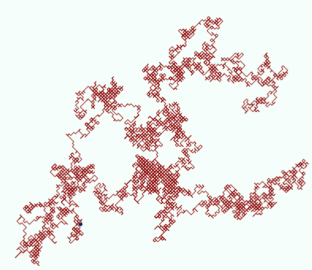
is one such ordering. This ordering has 3 runs of ones (in the first position, in the third through fourth positions, and in the final 4 positions).

Let the random variable R be the number of runs of ones. Find $\mathbb{E}[R]$.

(Hint: Write R as $R_1 + \cdots + R_{m+n}$, where R_i is defined to be 1 if a run begins at slot i and 0 otherwise.)

Problem 4

A particle begins at the origin at time 0. It repeatedly and independently chooses its steps uniformly at random from the set $\{(1,1), (1,-1), (-1,1), (-1,-1)\}$. (For example, after 2 steps the particle might be at the point $(2,0)$, perhaps after taking a $(1,-1)$ step and then a $(1,1)$ step.)



(i) Find the particle's expected squared distance from the origin after taking n steps.

(ii) Perform a Monte Carlo simulation to verify your result from part (i), for $n \in \{100, 1000, 10{,}000\}$.

Problem 5

Suppose that $\Omega = \{\mathrm{A}, \mathrm{B}, \mathrm{C}\} \times \mathbb{Z}$. We will represent an element of Ω as $\omega = (\omega_1, \omega_2)$, where $\omega_1 \in \{\mathrm{A}, \mathrm{B}, \mathrm{C}\}$ and $\omega_2 \in \mathbb{Z}$. Suppose that $\mathbb{P}(\omega_1 = \mathrm{A}) = \frac{1}{4}$, $\mathbb{P}(\omega_1 = \mathrm{B}) = \frac{1}{5}$, and $\mathbb{P}(\omega_1 = \mathrm{C}) = \frac{11}{20}$.

Suppose further that (i) the conditional distribution of ω_2 given $\{\omega_1 = \text{A}\}$ has probability mass function $n \mapsto \frac{1}{3}2^{-|n|}$, (ii) the conditional distribution of ω_2 given $\{\omega_1 = \text{B}\}$ is the uniform distribution on $\{-2, -1, 0, 1, 2\}$, and (iii) the conditional distribution of ω_2 given $\{\omega_1 = \text{C}\}$ has probability mass function $n \mapsto \mathbf{1}_{\{n \geq 1\}} \frac{6}{\pi^2 n^2}$.

Find the conditional distribution of ω_1 given:

(i) $\omega_2 = 2$
(ii) $\omega_2 = 7$
(iii) $\omega_2 \geq 10$

(In each case, express the conditional probabilities as percentages rounded to the nearest hundredth of a percent.)

Problem 6

Consider a random independent sequence of letters uniformly distributed in $\{\texttt{a}, \texttt{b}, \ldots, \texttt{z}\}$. Use Monte Carlo simulation to estimate the expected number of letters that appear in the sequence up to the first appearance of `aa`.

Repeat with `ab` in place of `aa`. Based on your findings, is the expected time to the first `aa` different from the expected time to the first `ab`?