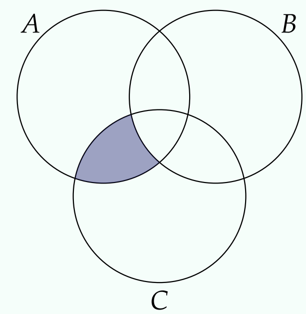
DATA 1010
Problem Set 4
Due 05 October 2018 at 11 PM

Problem 1



(a) Use set operations to express the shaded region in the Venn diagram in terms of A, B, and C.

(b) Suppose that $|A| = 5$, $|\Omega| = 10$, and $A \subset \Omega$. How many sets $B \subset \Omega$ satisfy the property that $A \cap B = \emptyset$?

(c) Find the least possible cardinality of a set which can be written as a disjoint union of 10 sets of pairwise unequal cardinalities.

Problem 2

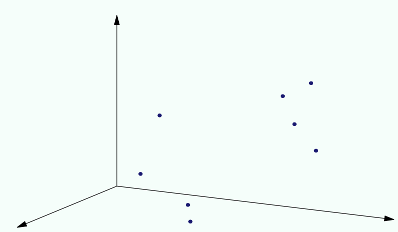
Write (from scratch) a function `binary` which interprets a string of 0's and 1's as the binary representation of an integer and returns the value of that integer.

```
binary("001001") == 9
binary("1001001011110101010") == 300970
```

Problem 3

The columns of the following matrix are plotted in the figure to the right.

```
1.25  0.75  0.25  1.25  1.5  1.5   0.75  0.75
2.25  1.0   1.75  1.25  1.0  2.25  1.75  0.75
1.25  0.0   0.5   0.25  0.5  1.0   1.0   0.75
```



Identify the plane passing through the origin whose sum of squared distances to these points is as small as possible.

Problem 4

Show that every representable `Float64` between 2^{-1021} and 2^{1022} can be doubled or halved with no roundoff error.

Problem 5

Recall that if A is an $m \times n$ matrix with rank n and $\mathbf{b} \in \mathbb{R}^m$, then the vector $\mathbf{x}$ which minimizes $|A\mathbf{x} - \mathbf{b}|^2$ is

$$\mathbf{x} = (A'A)^{-1}A'\mathbf{b}. \tag{5.1}$$

Also, recall that if U is a matrix with orthonormal columns, then the vector in the span of the columns of U is closest to $\mathbf{b}$ is given by $UU'\mathbf{b}$. Show that the formula $UU'\mathbf{b}$ can be obtained using (5.1).

Problem 6

Suppose that A is an invertible, symmetric matrix. Which of the following may we conclude? Select all that apply, and explain your reasoning.

(a) A is orthogonal.

(b) A^2 is symmetric.

(c) A^{-1} is symmetric.

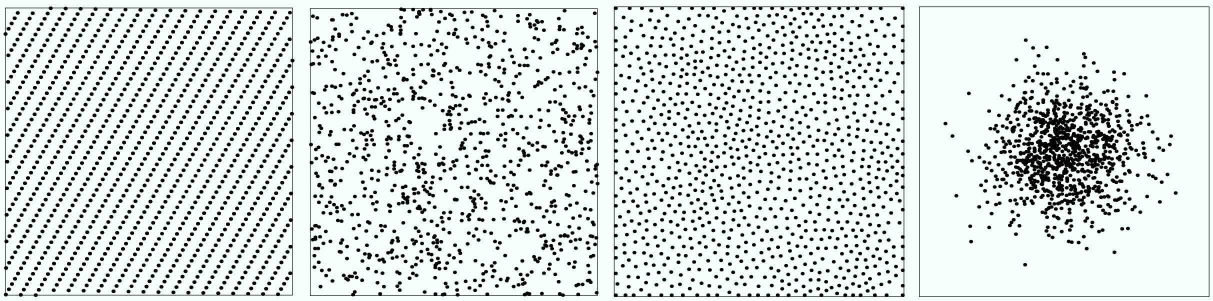
(d) $A - 3I$ is symmetric.

(e) A is orthogonally diagonalizable.

(f) The singular values of A are all positive.

Problem 7

One of the following three pictures was obtained by generating 2000 independent random numbers uniformly distributed in $[0, 1]$, arranging them into blocks of 2, and plotting the resulting 1000 ordered pairs in the plane. Which is it?



Hint: `using Plots; scatter(rand(1000),rand(1000);aspect_ratio=1)`

Problem 8

Show, using only the three basic properties of a probability space, that $\mathbb{P}(E \cup F \cup G) = \mathbb{P}(E) + \mathbb{P}(F) + \mathbb{P}(G)$ for any events E, F, and G which are pairwise disjoint.

Problem 9

Write a Julia program to find the exact value of the probability that in eight flips of a fair coin, no two consecutive flips turn up heads.

Two hints: (i) in class we wrote a recursive function to generate all possible outcomes for a similar random experiment, and (ii) if you store your flip sequences as strings, you can use the function `occursin` to check for the substring `HH`.

Problem 10

The 52 cards in a standard deck (13 spades, 13 clubs, 13 diamonds, 13 hearts), are dealt into four hands of 13 cards each. What is the probability that one of the hands contains all of the hearts?