DATA 1010
PROBLEM SET 2
DUE 21 SEPTEMBER 2018 AT 11 PM

Problem 1

Each of 68 people is interviewed and scored on a scale from 0 to 10 in three different categories. A composite score is obtained for each person by averaging the person's category scores. These data are arranged into a 68×4 matrix X, so that each row consists of a particular interviewee's category scores and composite score.

Find the determinant of $X'X$.

Problem 2

Use matrix differentiation to find the vector $\mathbf{x} \in \mathbb{R}^n$ which minimizes the expression $|W(A\mathbf{x} - \mathbf{b})|^2$, where A is an $m \times n$ matrix and W is an $m \times m$ matrix. You may assume that WA is full-rank.

Problem 3

Find the derivative of $|\mathbf{x}|$ with respect to $\mathbf{x}$. Hint: write $|\mathbf{x}|$ as $\sqrt{\mathbf{x}'\mathbf{x}}$ and use the chain rule, which says that if $g : \mathbb{R}^n \to \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$, then

$$\frac{\partial}{\partial \mathbf{x}} f(g(\mathbf{x})) = f'(g(\mathbf{x})) \frac{\partial g}{\partial \mathbf{x}}(\mathbf{x}).$$

Interpret your answer geometrically and explain why it makes sense.

Problem 4

(i) Find the line through the origin for which the sum of squared distances from the line to points in the set

$$\{(3,-1),(2,4),(-1,-1),(-2,2),(-3,1),(5,-1),(-2,4)\}$$

is as small as possible.

(ii) Find the slope of the zero-intercept line of best fit for these points using the formula $m = (A'A)^{-1}A'\mathbf{b}$, where A is a column vector whose entries are the x coordinates of the points and where $\mathbf{b}$ is a column vector whose components are the y-components of the points (in the same order). Recall that this is the line which minimizes $\sum_i (mx_i - y_i)^2$ where (x_i, y_i) ranges over the given points.

(iii) Draw both of these lines and explain why they are not the same even though they both minimize a sum of squared distances.

```
using Plots, LinearAlgebra
A = [3 2 -1 -2 -3 5 -2; -1 4 -1 2 1 -1 4]
scatter(A[1,:], A[2,:])
```

Problem 5

Find a value of x which is less than 1 and for which `1 + x + x + x > 1 + 3x` returns `true`. Explain this behavior.

Problem 6

Explain why the following function returns a value rather than running forever. Explain why it returns the particular value that it returns.

```
function countdown()
    x = 1.0
    ctr = 0
    while x > 0.0
        x /= 2
        ctr += 1
    end
    ctr
end
```

Problem 7

Show that an invertible, square matrix and its inverse have the same condition number.

Problem 8

Consider the $n \times n$ Frank matrix F_n, defined as shown in the code block below.

```
function frankmatrix(n)
    A = zeros(n,n)
    for i=1:n
        for j=1:n
            if j == i-1
                A[i,j] = n + 1 - i
            elseif j ≥ i
                A[i,j] = n + 1 - j
            end
        end
    end
    A
end
```

Find $F_n^{-1}\mathbf{v}$, where $\mathbf{v} \in \mathbb{R}^n$ has all components equal to 1, by inspection. (Generate F_n for some small values of n and look at it).

Evaluate `frankmatrix(n) \ ones(n)` for $n \in \{10, 15, 20, 25, 30\}$ and calculate the norm of the difference between this numerical solution and the true solution. Compare your result to the product of `eps()` (which equals 2^{-52}, the gap between 1 and the nearest representable 64-bit floating point) and the condition number of F_n (which can be calculated using the function `cond`). Hint: a good way to do this comparison is to plot the log of each of these quantities over the specified range of n values.

Based on your findings, comment on whether the algorithm being used for `\` is stable.

Problem 9

Consider the following PRNG (which was actually widely used in the early 1970s): we begin with an odd positive integer a_1 less than 2^{31} and for all $n \geq 2$, we define a_n to be the remainder when dividing $65539a_{n-1}$ by 2^{31}.

Use Julia to calculate $9a_{3n+1} - 6a_{3n+2} + a_{3n+3}$ for the first 10^6 values of n, and show that there are only *15* unique values in the resulting list (!). Explain what you would see if you plotted many points of the form $(a_{3n+1}, a_{3n+2}, a_{3n+3})$ in three-dimensional space.