DATA 1010
In-class exercises
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Problem 1

Suppose that Z has density $f_Z(z) = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-z^2/2}$. Find the density of $X = \sigma Z + \mu$.

Solution

We have $\sigma Z + \mu \in (x, x + \mathrm{d}x)$ if and only if $Z \in \left(\frac{x-\mu}{\sigma}, \frac{x-\mu}{\sigma} + \frac{\mathrm{d}x}{\sigma}\right)$, and that happens with probability $f_Z\left(\frac{x-\mu}{\sigma}\right)\frac{\mathrm{d}x}{\sigma}$, so we have

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}}\mathrm{e}^{-\frac{(x-\mu)^2}{2\sigma^2}}.$$

Problem 2

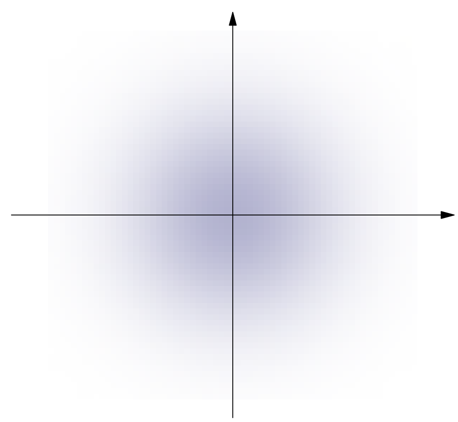
Suppose that $Z_1, \ldots, Z_n$ are independent standard normal random variables. Find the density of the random vector $\mathbf{Z} = [Z_1, \ldots, Z_n]$, and show that it is rotationally symmetric.

Solution

Since the random variables are independent, we multiply their densities:

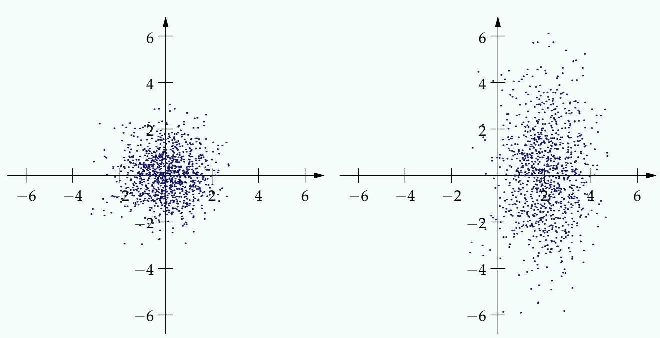
$$f_{\mathbf{Z}}(\mathbf{z}) = \frac{1}{(2\pi)^{n/2}}\mathrm{e}^{-(z_1^2+\cdots+z_n^2)/2}.$$

This function is rotationally symmetric since $f_{\mathbf{Z}}(\mathbf{z}) = f_{\mathbf{Z}}(\mathbf{w})$ whenever $\mathbf{z} = \mathbf{w}$. A heatmap of the density is shown in the figure.



Problem 3

Find the function which maps the point cloud on the left to the point cloud on the right.



Solution

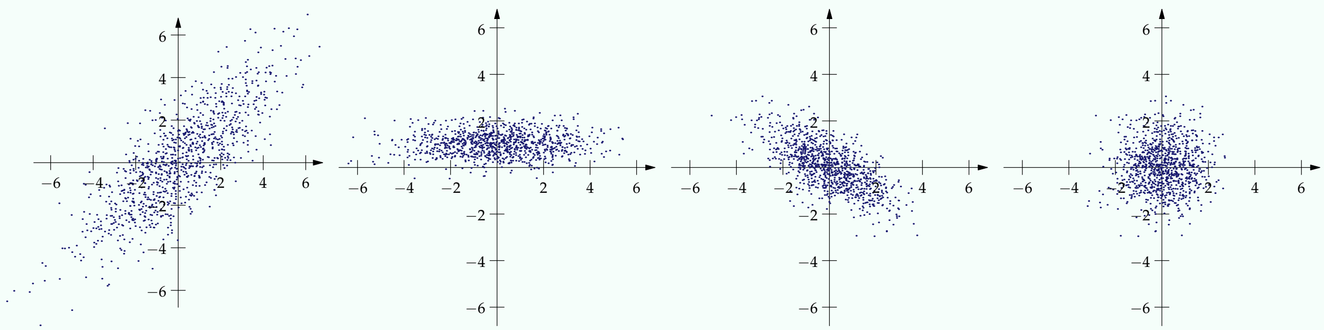
The function scales by a factor of 2 in the vertical direction and shifts the cloud two units to the right. Therefore, the function is $(x, y) \mapsto (x + 2, 2y)$.

We can represent this affine function in matrix form as $\mathbf{x} \mapsto A\mathbf{x} + \mathbf{b}$, where $A = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix}\right]$ and $\mathbf{b} = \left[\begin{smallmatrix} 2 \\ 0 \end{smallmatrix}\right]$.

Problem 4

For each $i \in \{1, 2, 3\}$, the ith point cloud below is obtained by sampling 1000 times from $A_i Z + \mu_i$, where Z is a vector of two independent normal random variables, A_i is a 2×2 matrix of constants, and μ_i is a constant vector in $\mathbb{R}^2$. Approximate (A_i, μ_i) for each $i \in \{1, 2, 3\}$. (Note: A_i is not uniquely determined, so just find an A_i that works.)

For reference, a plot of 1000 independent samples from Z is shown in the fourth figure.



Solution

To map the point cloud in the fourth figure to a point cloud which has approximately the same shape and location as the first figure, we apply a linear transformation which maps the standard basis vectors in $\mathbb{R}^2$ to the major and minor semiaxes of the ellipse.

```
using Plots
A = [2 2; 1 -1]'
μ = [0,1]
points = [A*randn(2)+μ for i=1:1000]
scatter([x for (x,y) in points],[y for (x,y) in points],aspect_ratio=:equal,legend=false)
```

For A_2 we use `[2 0; 0 1]'`, and for A_3 we use `[1 -1; 1/2 1/2]'`. The only nonzero center is μ_2, which should be `[0,1]`.

Problem 5

Find an expression for the probability density function of AZ, where Z is a vector of n independent standard normal random variables and A is an invertible $n \times n$ matrix.

Solution

If we consider a small patch $\mathrm{d}R$ of $\mathbb{R}^2$ with area $\mathrm{d}x\,\mathrm{d}y$ and containing a point $\mathbf{x} = [x_1, \ldots, x_n]$, then the probability that $A\mathbf{Z} \in \mathrm{d}R$ is equal to the probability that $\mathbf{Z} \in A^{-1}\mathrm{d}R$. Since $\mathbf{Z}$ has density

$$f_{\mathbf{Z}}(\mathbf{z}) = \frac{1}{(2\pi)^{n/2}} \mathrm{e}^{-(z_1^2 + \cdots + z_n^2)/2} = \frac{1}{(2\pi)^{n/2}} \mathrm{e}^{-\mathbf{z}'\mathbf{z}/2},$$

where $\mathbf{z} = [z_1, \ldots, z_n]$, we have

$$\mathbb{P}(\mathbf{Z} \in A^{-1}\mathrm{d}R) = f_{\mathbf{Z}}(A^{-1}(\mathbf{x}))\,\mathrm{area}(A^{-1}\mathrm{d}R).$$

Since $\det(A)$ is the factor by which A transforms volumes, we have $\text{area}(A^{-1}\mathrm{d}R) = \det(A)^{-1}\,\mathrm{d}x\,\mathrm{d}y$. Therefore,

$$\mathbb{P}(\mathbf{Z} \in A^{-1}\mathrm{d}R) = \frac{1}{(2\pi)^{n/2}\det A}\exp\left(-\frac{1}{2}\mathbf{x}'(A^{-1})'A^{-1}\mathbf{x}\right)\,\mathrm{d}x\,\mathrm{d}y.$$

So the density of X is

$$f_{\mathbf{Z}}(\mathbf{z}) = \frac{1}{(2\pi)^{n/2}\det A}\exp\left(-\frac{1}{2}\mathbf{x}'(A^{-1})'A^{-1}\mathbf{x}\right).$$

Problem 6

Find the covariance matrix Σ of $A\mathbf{Z}$. Express the density of $A\mathbf{Z} + \mu$ in terms of Σ, where μ is a constant vector.

Solution

The covariance matrix of a random vector Y is $\mathbb{E}[YY']$, so we have

$$\mathbb{E}[(AZ)(AZ)'] = \mathbb{E}[AZZ'A'] = A\mathbb{E}[ZZ']A' = AA'.$$

Since $\det(AA') = \det(\Sigma) \implies (\det A)^2 = \det\Sigma$ and $(A^{-1})'A^{-1} = (AA')^{-1} = \Sigma^{-1}$, we have

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{1}{\sqrt{(2\pi)^n\det\Sigma}}\exp\left(-\frac{1}{2}\mathbf{x}'\Sigma^{-1}\mathbf{x}\right).$$