DATA 1010
In-class exercises
Samuel S. Watson
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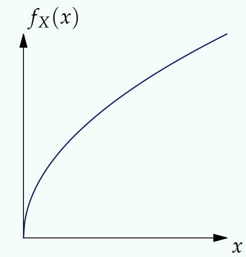
Problem 1

Find the expected value of a random variable whose probability density function is $f(x) = c\sqrt{x}\mathbf{1}_{0\leq x\leq 1}$ for some constant c.



Solution

We know that $\int_0^1 c\sqrt{x}\,\mathrm{d}x = 1$ since the total mass of the distribution must be 1, and this implies that $\frac{2}{3}c = 1 \implies c = \frac{3}{2}$.
Then the expectation is given by

$$\int_0^1 x\left(\frac{3}{2}\sqrt{x}\right)\mathrm{d}x = \frac{3}{5}.$$

Problem 2

Find the PDF of the distribution of X if the joint distribution of X and Y is $f_{X,Y}(x,y) = e^{-x-y}\mathbf{1}_{x\geq 0}\mathbf{1}_{y\geq 0}$.

Solution

If $A \subset [0,\infty)$, then the probability that $X \in A$ is equal to

$$\int_A\int_0^1 \mathrm{e}^{-x-y}\,\mathrm{d}y\,\mathrm{d}x = \int_A \mathrm{e}^{-x}\,\mathrm{d}x.$$

Therefore, the density function of the distribution of X is e^{-x}.

We can see that finding marginal distributions from joint distributions works analogously to the discrete case: rather than summing along horizontal and vertical lines, we *integrate* along horizontal and vertical lines.

Problem 3

Suppose that T is the triangle with vertices at the origin, $(0,1)$, and $(1,0)$. Suppose that X and Y have joint density function proportional to xy on T (and zero elsewhere). Find the conditional density of Y given X. Are X and Y independent?

Solution

First we find the marginal distribution of X:

$$f_X(x) = \int_0^{1-x} xy\,\mathrm{d}y = \frac{x(1-x)^2}{2}.$$

Then the conditional density of Y given X is

$$f_{Y\,|\,X=x}(y) = \frac{xy}{x(1-x)^2/2}\mathbf{1}_{\{(x,y)\in T\}} = \frac{2xy}{x(1-x)^2}\mathbf{1}_{\{y\leq 1-x\}}.$$

We can see that X and Y are not independent, since the joint distribution of Y given $X = x$ does depend on x.

Problem 4

Find the expectation of XY, where X and Y are random variables whose joint distribution is uniform on the set of points which are in the unit disk and between the positive x-axis and the ray $\theta = \pi/4$.

Solution

Each small patch of area $\mathrm{d}A$ around a point (x, y) in the given region contributes $xy\,dA$ to the expectation of the random variable XY. Therefore, to find the expectation we total up (that is, integrate) these quantities over the region. Since the region is bounded by rays and an arc of an origin-centered circle, we use polar coordinates:

$$\int_0^{\pi/4} \int_0^1 (r\cos\theta)(r\sin\theta) r \,\mathrm{d}r\,\mathrm{d}\theta = \frac{1}{16},$$

where we performed the integration using Wolfram Alpha. Or, if you want to do it in Julia:

Julia

```
using SymPy
@vars r θ
integrate(integrate(r*cos(θ)*r*sin(θ)*r,(r,0,1)),(θ,0,PI/4))
```

The generalization of the idea we are using in this solution is that if X and Y have joint pdf f and if $g : \mathbb{R}^2 \to \mathbb{R}$, then $\mathbb{E}[g(X, Y)] = \int_{\mathbb{R}^2} gf$.

Problem 5

Write an expression for the probability of getting exactly k heads when flipping a p-coin n times. (Note: a p-coin is a coin with probability p of turning up heads on any given flip.)

Plot the resulting expression for a variety of values of n, k, and p.

Solution

The probability of getting k heads followed by $n-k$ tails is $p^k(1-p)^{n-k}$. In fact, for any arrangement of k H's and $n-k$ T's, the probability of getting that particular sequence is $p^k(1-p)^{n-k}$. To find the total probability mass associated with all of these outcomes, we sum all of these masses. Since they're equal and there are $\binom{n}{k}$ of them, we end up with

$$\binom{n}{k} p^k (1-p)^{n-k}.$$

Directly using this formula is not a good way to compute these probabilities, because $\binom{n}{k}$ overflows and $p^k(1-p)^{n-k}$ underflows (consider, for example, $p = 0.5$ and $n \geq 1075$—then $p^k(1-p)^{n-k}$ would round to 0.0 in `Float64` arithmetic). A better way is to take the logarithm of this expression and use the special function `lgamma(n)`, which directly calculates the logarithm of $(n+1)!$.

Julia

```
using Plots, SpecialFunctions
pgfplots()
binom(n,k,p) = exp(lgamma(n+1) - lgamma(k+1) - lgamma(n-k+1)
                   + k*log(p) + (n-k) * log(1-p))
n = 10
p = 0.5
scatter(0:n,[binom(n,k,p) for k=0:n])
```