DATA 1010
In-class exercises
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Problem 1

Consider a pair of random variables (X, Y) whose joint distribution is supported on $[0,1]^2$ with density $6x^2y$. Show that X and Y are independent.

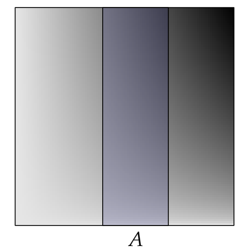
Now suppose the joint density is $\frac{3}{2}(x^2 + y^2)$. Show that X and Y are not independent.

Solution

For any $A \subset [0,1]$, we have

$$\mathbb{P}(X \in A) = \int_{A \times [0,1]} 6x^2 y \, \mathrm{d}x \, \mathrm{d}y = \int_A 3x^2 \, \mathrm{d}x,$$

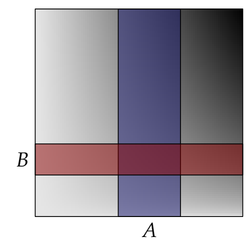
so the density of the distribution of X is $3x^2$ on $[0,1]$. Similarly, the distribution of Y has density $2y$ on $[0,1]$.



For any $A, B \subset [0,1]$, we have

$$\mathbb{P}(\{X \in A\} \cap \{Y \in B\}) = \int_{A \times B} 6x^2 y \, \mathrm{d}x \, \mathrm{d}y = \left(\int_A 3x^2 \, \mathrm{d}x \right) \left(\int_B 2y \, \mathrm{d}y \right) = \mathbb{P}(X \in A)\mathbb{P}(Y \in B),$$

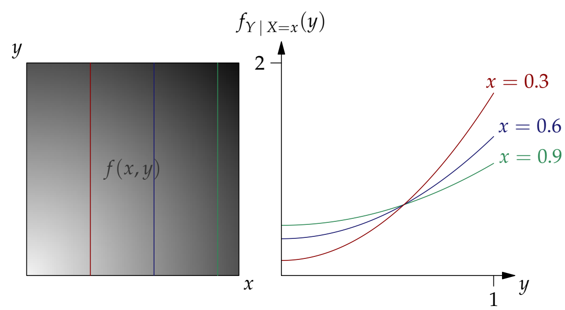
as desired.



For $(x, y) \mapsto \frac{3}{2}(x^2 + y^2)$, we take the conditioning perspective on independence: two random variables X and Y are independent if the conditional distribution of Y given $\{X = x\}$ is the same as the distribution of Y (in other words, knowing X doesn't tell us anything about Y).

If we slice the graph of the density function $\frac{3}{2}(x^2 + y^2)$ along the vertical line at position x, then the conditional density of Y is

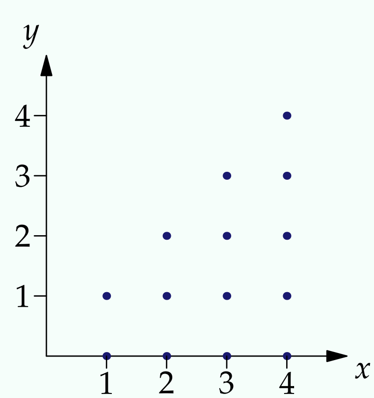
$$\frac{\frac{3}{2}(x^2 + y^2)}{\int_0^1 \frac{3}{2}(x^2 + y^2) \, \mathrm{d}y} = \frac{3x^2 + 3y^2}{3x^2 + 1}.$$



This function is different for different x values, as shown in the figure. Therefore, the two random variables are not independent.

Problem 2

Consider a pair of random variables X and Y with joint distribution m, where m is the probability mass function shown. Find the conditional distribution of Y given $X = x$ for each value of x.



Solution

The conditional expectation of Y given $X = 1$ is $\frac{1}{2}$, since the conditional distribution of Y given $X = 1$ is uniform on $\{0, 1\}$. Similarly, given $X = 2$, the conditional expectation of Y is 1, and so on. In general, we can say that the conditional expectation of Y given X is $X/2$.

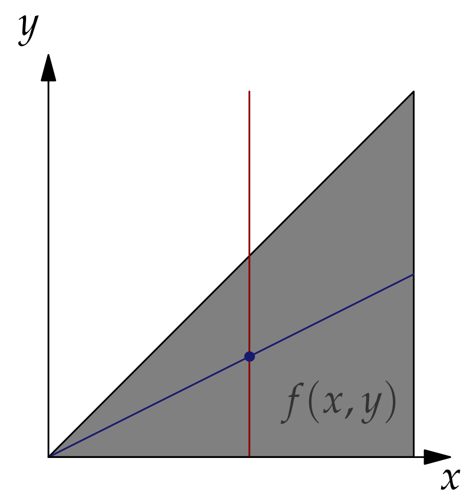
Problem 3

Suppose that f is the function which returns 2 for any point in the triangle with vertices $(0,0)$, $(1,0)$, and $(0,1)$ and otherwise returns 0. If (X,Y) has joint PDF f, find the conditional expectation of Y given $\{X = x\}$.

Solution

Conditioning on $X = x$ amounts to restricting the sample space to the points which map to the vertical line at position x. Slicing the joint density along that line, we find that Y is uniformly distributed on the interval $[0, x]$. The mean of the uniform distribution on $[0, x]$ is $x/2$, so we say that $\mathbb{E}[Y \mid X = x] = x/2$.

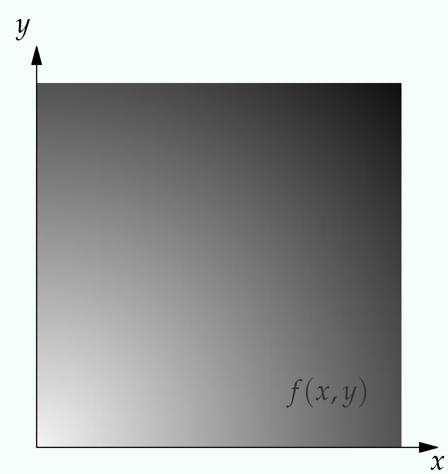
We can express this more succinctly by saying $\mathbb{E}[Y \mid X] = X/2$.



Problem 4

Given that X and Y have joint PDF $f(x, y) = \frac{3}{2}(x^2 + y^2)$ on $[0, 1]^2$, find the conditional expectation of Y given X.

Begin by sketching an estimate of the conditional expectation on the graph shown.



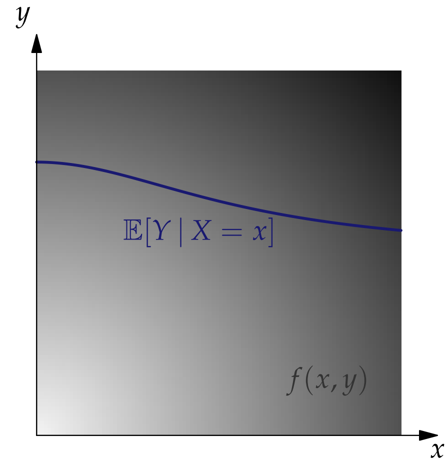
Solution

The conditional density of Y given $X = x$ is

$$f_{Y\,|\,\{X=x\}}(y) = \frac{3x^2 + 3y^2}{3x^2 + 1},$$

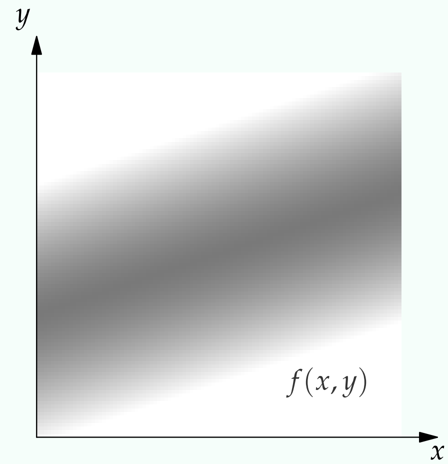
as we figured out in Problem 1. To find the mean of this density, we multiply by y and integrate to get

$$\mathbb{E}[Y \mid X = x] = \int_0^1 y f_{Y\,|\,\{X=x\}}(y)\,\mathrm{d}y = \frac{3\,(2x^2+1)}{4\,(3x^2+1)}.$$



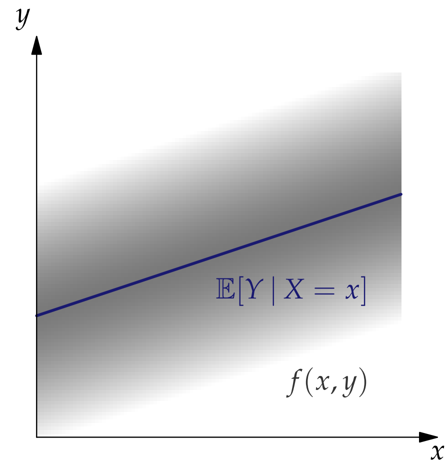
Problem 5

Given that X and Y have joint PDF shown in the figure, sketch an estimate of the conditional expectation of Y given $X = x$.



Solution

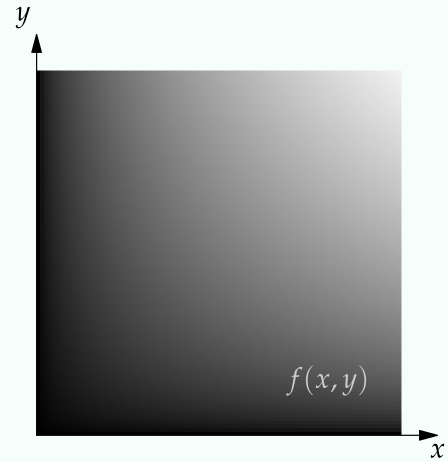
Along any vertical line, the mass density is distributed symmetrically around a point (which increases linearly as x increases), so the conditional expectation of Y at that x-value is the y-coordinate of that symmetry point.



Problem 6

Given that X and Y have joint PDF $f(x,y) = \frac{9}{5}(1-\sqrt{xy})$ on $[0,1]^2$, find the conditional expectation of Y given X.

Begin by sketching an estimate of the conditional expectation on the graph shown.



Solution

The conditional density of Y given $X = x$ is

$$f_{Y\,|\,\{X=x\}}(y) = \frac{1-\sqrt{xy}}{\int_0^1 1-\sqrt{xy}\,\mathrm{d}y} = \frac{1-\sqrt{xy}}{1-2\sqrt{x}/3}.$$

To find the mean of this density, we multiply by y and integrate to get

$$\mathbb{E}[Y\,|\,X=x] = \int_0^1 y f_{Y\,|\,\{X=x\}}(y)\,\mathrm{d}y = \frac{3\left(4\sqrt{x}-5\right)}{10\left(2\sqrt{x}-3\right)}.$$