DATA 1010
In-class exercises
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Problem 1

Consider the random variable 1_E which maps each $\omega \in E$ to 1 and each $\omega \in E^c$ to 0. Find the expected value of 1_E.

Solution

We have $\mathbb{E}[1_E] = 1 \cdot \mathbb{P}(E) + 0 \cdot \mathbb{P}(E^c) = \mathbb{P}(E)$.

We can see that random variables *generalize* events: if we identify an event E with its indicator random variable 1_E, then we can perform set operations (for example, intersection of two sets corresponds to multiplication of their indicator random variables), and we can take probabilities of events (by finding the expectations of the corresponding indicator random variables).

This makes sense from the random-experiment point of view: An event is a statement which is either true (1) or false (0) for each outcome. A random variable *also* associates a number to each outcome, but the values are not restricted to $\{0, 1\}$.

Problem 2

Find the expected value of $X + Y$, where X and Y are independent random variables whose distributions have constant probability mass functions on $\{0, 1, 2, 3\}$. What is the relationship between $\mathbb{E}[X + Y]$, $\mathbb{E}[X]$, and $\mathbb{E}[Y]$?

Solution

We have already calculated the probability mass function, so we can calculate the expected value directly from that. We get

$$0 \cdot \frac{1}{16} + 1 \cdot \frac{2}{16} + 2 \cdot \frac{3}{16} + 3 \cdot \frac{4}{16} + 4 \cdot \frac{3}{16} + 5 \cdot \frac{2}{16} + 6 \cdot \frac{1}{16} = 3.$$

We can see that $\mathbb{E}[X + Y] = 1.5 + 1.5 = \mathbb{E}[X] + \mathbb{E}[Y]$.

Problem 3

Consider a uniformly random permutation of the integers $\{1, 2, 3, 4\}$. Let X be 3 if the first two numbers are 1 and 2, and let it be 2 otherwise. Let Y be 6 if the second and third numbers are 2 and 3; let it be 4 if the second number is 2 and the third number is not 3; and let it be 0 if the second number is not 2.

Find $\mathbb{E}[X + Y]$, $\mathbb{E}[X]$, and $\mathbb{E}[Y]$.

Solution

The probability that the first two numbers are 1 and 2 is $\frac{2}{24}$. So the expected value of X is

$$\mathbb{E}[X] = 3 \cdot \frac{2}{24} + 2 \cdot \frac{22}{24} = \frac{50}{24}$$

The expected value of Y is

$$\mathbb{E}[Y] = 6 \cdot \frac{2}{24} + 4 \cdot \frac{4}{24} + 0 \cdot \frac{18}{24} = \frac{28}{24}.$$

To find the expected value of $X + Y$, we enumerate the elements of the sample space and find the value of $X + Y$ for each one:

$$
\begin{aligned}
(1,2,3,4) &: 3+6=9 \\
(1,2,4,3) &: 3+4=7 \\
(1,4,2,3) &: 2+0=2 \\
(4,1,2,3) &: 2+0=2 \\
(1,3,2,4) &: 2+0=2 \\
(1,3,4,2) &: 2+0=2 \\
(1,4,3,2) &: 2+0=2 \\
(4,1,3,2) &: 2+0=2 \\
(3,1,2,4) &: 2+0=2 \\
(3,1,4,2) &: 2+0=2 \\
(3,4,1,2) &: 2+0=2 \\
(4,3,1,2) &: 2+0=2 \\
(2,1,3,4) &: 2+0=2 \\
(2,1,4,3) &: 2+0=2 \\
(2,4,1,3) &: 2+0=2 \\
(4,2,1,3) &: 2+4=6 \\
(2,3,1,4) &: 2+0=2 \\
(2,3,4,1) &: 2+0=2 \\
(2,4,3,1) &: 2+0=2 \\
(4,2,3,1) &: 2+6=8 \\
(3,2,1,4) &: 2+4=6 \\
(3,2,4,1) &: 2+4=6 \\
(3,4,2,1) &: 2+0=2 \\
(4,3,2,1) &: 2+0=2
\end{aligned}
$$

So the expected value of $X+Y$ is $\frac{1}{24}(9+8+7+6+6+6+2\cdot 18) = \frac{78}{24}$. This is the same as $\mathbb{E}[X+Y]$. Furthermore, our approach to this problem explains *why* $\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y]$: we can get $\mathbb{E}[X]$ by multiplying each entry in the first column by $m(\omega) = \frac{1}{24}$ and then summing down the column. We get $\mathbb{E}[Y]$ similarly using the second column. Finally, $\mathbb{E}[X+Y]$ is obtained by summing these same quantities, just along rows first.

Problem 4

Consider two unfair coins: the first has probability $\frac{3}{5}$ of coming up heads, and the second has probability $\frac{1}{3}$ of coming up heads. We define X to be 2 if the first coin is heads and -1 otherwise, and we define Y to be 1 if the first coin is tails and the second coin is heads, and 2 otherwise.

Show that $\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y]$.

Solution

Let's enumerate the elements of Ω and work out the contribution of each one to each of the expectations $\mathbb{E}[X], \mathbb{E}[Y]$, and $\mathbb{E}[X+Y]$:

	$m(\omega)$	$X(\omega)$	$Y(\omega)$	$X(\omega)m(\omega)$	$Y(\omega)m(\omega)$
(H,H)	$\frac{1}{3}\cdot\frac{3}{5}$	2	2	$\frac{1}{5}\cdot 2$	$\frac{1}{5}\cdot 2$
(T,H)	$\frac{2}{3}\cdot\frac{3}{5}$	-1	1	$\frac{2}{5}\cdot -1$	$\frac{2}{5}\cdot 1$
(H,T)	$\frac{1}{3}\cdot\frac{2}{5}$	2	2	$\frac{2}{15}\cdot 2$	$\frac{2}{15}\cdot 2$
(T,T)	$\frac{2}{3}\cdot\frac{2}{5}$	-1	2	$\frac{4}{15}\cdot -1$	$\frac{4}{5}\cdot 2$

Then $\mathbb{E}[X]$ is obtained by summing the entries of the fourth column, $\mathbb{E}[Y]$ the fifth, and $\mathbb{E}[X+Y]$ by adding the fourth and fifth columns and summing the entries of the result. Therefore, $\mathbb{E}[X]+\mathbb{E}[Y]$ is the sum of the same 8 numbers as $\mathbb{E}[X+Y]$, and the quantities are therefore equal.

Problem 5

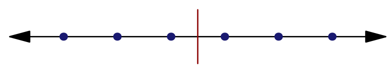
Suppose a fair coin is flipped and a fair die is rolled. We let X be 1 or 2 if the coin turns up tails or heads, respectively, and we let Y be the value of the die roll.

(a) Calculate $\mathbb{E}[XY]$ as well as $\mathbb{E}[X]$ and $\mathbb{E}[Y]$. What is the relationship between these three?

(b) Suppose that $Z = X + Y$ and calculate $\mathbb{E}[Z]$ and $\mathbb{E}[ZY]$. Do $\mathbb{E}[ZY]$, $E[Z]$, and $\mathbb{E}[Y]$ have the same relationship you found in (a)?

Solution

(a) We have $\mathbb{E}[X] = 1.5$ and $\mathbb{E}[Y] = 3.5$, either by direct calculation or by observing that these distributions are symmetric, so their means are at their respective points of symmetry:



To find the expected value of the product, we find the value of the product on each outcome:

ω	$X(\omega)$	$Y(\omega)$
(H,1)	2	1
(H,2)	2	2
(H,3)	2	3
(H,4)	2	4
(H,5)	2	5
(H,6)	2	6
(T,1)	1	1
(T,2)	1	2
(T,3)	1	3
(T,4)	1	4
(T,5)	1	5
(T,6)	1	6

Calculating $m(\omega)X(\omega)Y(\omega)$ for each row and summing, we get $\mathbb{E}[XY] = \frac{63}{12} = \frac{21}{4}$. Note that this is equal to $\mathbb{E}[X]\mathbb{E}[Y] = \frac{3}{2}\cdot\frac{7}{2}$.

One way of seeing why these worked out to be the same is to note that the sum $\sum_{\omega\in\Omega} m(\omega)X(\omega)Y(\omega)$ that we calculated can also be written as

$$\frac{1}{12}(1+2)(1+2+3+4+5+6) = \left[\frac{1}{2}(1+2)\right]\left[\frac{1}{6}(1+2+3+4+5+6)\right].$$

(b)

Problem 6

Shuffle a standard 52-card deck, and let X be the number of consecutive pairs of cards in the deck which are both red. Find $E[X]$.

Write some code to simulate this experiment and confirm that your answer is correct. Hint: store the deck of undrawn cards as a `Set`, and `pop!` cards from it as you draw. You can draw a random element from a set `S` using `rand(S)`.

Solution

Calculating $\mathbb{E}[X]$ from the distribution of X would be very complicated, because the distribution of X is complicated. Try to calculate, for instance, $\mathbb{P}(X = 10)$.

However, finding the mean is nevertheless manageable: we can write $X = X_2 + \ldots + X_{52}$ where X_j is equal to 1 if the cards in positions $j - 1$ and j are both red and is equal to 0 otherwise. We see that $\mathbb{E}[X_j] = (1/2)(25/51)$, since card j is red with probability 1/2, and card $j - 1$ is red with conditional probability 25/51, given that card j is red. So $\mathbb{E}[X] = \mathbb{E}[X_2] + \cdots + \mathbb{E}[X_{52}] = 51(1/2)(25/51) = 25/2$.

To simulate this experiment, we identify the cards using integers 1 to 52. We treat even numbers as red cards and odd numbers as black cards.

```julia
using Statistics
"""
Put the integers 1 to n in random order
"""
function draw(n)
    deck = Set(1:n)
    drawn_cards = Int64[]
    while !isempty(deck)
        card = rand(deck)
        push!(drawn_cards,card)
        pop!(deck,card)
    end
    drawn_cards
end

"""
Count the number of consecutive pairs of even numbers
in the sequence `cards`
"""
function countpairs(cards)
    pairs = 0
    for i=2:length(cards)
        if iseven(cards[i]) && iseven(cards[i-1])
            pairs += 1
        end
    end
    pairs
end
```