DATA 1010
In-class exercises
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Problem 1

A problem on a test requires students to match molecule diagrams to their appropriate labels. Suppose there are three labels and three diagrams and that a student guesses a matching uniformly at random. Let X denote the number of diagrams the student correctly labels.

(a) What is the probability mass function of the conditional distribution of X given the event $X \geq 1$?

(b) What is the probability mass function of the conditional distribution of X given the event that the student knows the first matching and has to guess at the other two?

Solution

(a) We set the $1/3$ unit of mass at 0 to 0, since we know that none of the outcomes which X maps to that value occur, and we gross up the remaining masses by a factor of $\frac{1}{2/3} = \frac{3}{2}$. So we get a mass of $\frac{3}{4}$ at 1 and a mass of $\frac{1}{4}$ at 3.

Note that we could have applied the conditioning on the sample space side (zeroing out the probability masses of the two ω's that map to 0 under X). The distribution of X under this measure is the same distribution we got by applying the conditioning procedure directly to the distribution of X.

(b) If one of the answers is known, then there are two equally likely ways to fill out the remaining two: correctly and incorrectly. Therefore, we get a probability mass of $\frac{1}{2}$ at 1 and a probability mass of $\frac{1}{2}$ at 3.

In probability space terms (assuming that the correct order is ABC), in (a) we are conditioning on the event $\{\text{ACB}, \text{CBA}, \text{BAC}\}$. Meanwhile, in (b) we are conditioning on the event $\{\text{ABC}, \text{ACB}\}$.

Problem 2

Consider the following experiment: we roll a die, and if it shows 2 or less we select Urn A, and otherwise we select Urn B. Next, we draw a ball uniformly at random from the selected urn. Urn A contains one red and one blue ball, while urn B contains 3 blue balls and one red ball.

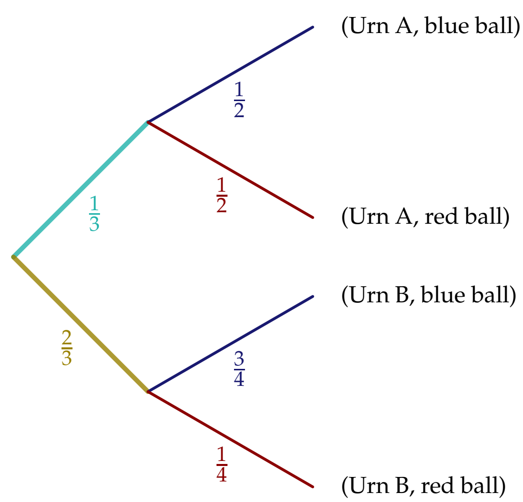
Find a probability space Ω which models this experiment, find a pair of events E and F such that $\mathbb{P}(E \mid F) = \frac{3}{4}$.

Solution

The four possible outcomes of this experiment are (A, blue), (A, red), (B, blue), and (B, red). So we let our probability space Ω consist of those four outcomes.

The probability of the outcome (A, blue) is equal to the probability that Urn A is selected times the conditional probability of selecting a blue ball given that Urn A was selected. We interpret the information that Urn A contains an equal number of blue and red balls as a statement that this conditional probability should be $\frac{1}{2}$. Therefore, we assign the probability $\frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$ to the event (A, blue).

Likewise, the probabilities we assign to the three other outcomes are $\frac{1}{6}$, $\frac{1}{2}$, and $\frac{1}{6}$, respectively.



With probabilities thus assigned to the outcomes in Ω, we should have $\mathbb{P}(E \mid F) = \frac{3}{4}$ where E is the event that we select a blue ball and F is the event that Urn B was selected. Let us check that this is indeed the case:

$$\frac{\mathbb{P}(E \cap F)}{\mathbb{P}(F)} = \frac{\frac{1}{2}}{\frac{2}{3}} = \frac{3}{4}.$$

We have arrived at an important insight: a probability space may alternatively by specified via a tree diagram showing conditional probabilities, or by the probability space Ω consisting of the endpoints of the tree diagram. We can translate back and forth between these two representations by multiplying along branches to get from the tree's conditional probabilities to Ω's outcome probabilities or by calculating conditional probabilities to go from Ω to the tree diagram.

Solution

We have already calculated the probability mass function, so we can calculate the expected value directly from that.

$$0 \cdot \frac{1}{16} + 1 \cdot \frac{2}{16} + 2 \cdot \frac{3}{16} + 3 \cdot \frac{4}{16} + 4 \cdot \frac{3}{16} + 5 \cdot \frac{2}{16} + 6 \cdot \frac{1}{16} = 3.$$

Problem 3

Find the maximum possible value of $\frac{|A\mathbf{x}|}{|\mathbf{x}|}$ where $\mathbf{x} \in \mathbb{R}^3$ and

$$A = \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix}.$$

Solution

We find the norm of the leading left singular vector, which is $6\sqrt{10} \approx 18.97$.

Problem 4

Express the largest representable `Float64` in base-10 scientific notation, accurate to 3 decimal places. Express the smallest positive representable Float64 in base-10 scientific notation, accurate to 3 decimal places.

Solution

The largest representable `Float64` is

```
2.0^1023 + (2.0^1023 - 2.0^971)
```

which is 1.798×10^{308}, to four significant figures.

The least representable `Float64` is `0.5^1074`, which is 4.941×10^{-324} to four significant figures.

Note that we cannot find the first value by calculating `2.0^1074 - 2.0^971`, because the first value overflows (that is, it becomes `Inf`). Similarly, we can't calculate the smallest float using `1/2.0^1074`, since the denominator evaluates to `Inf`.

Problem 5

Suppose that A is a matrix with the property that each column has norm 3 and every pair of distinct columns has angle 60 degrees between them. Find $A'A$.

Solution

The entries of $A'A$ are the pairwise dot products of the columns of A. Therefore, every diagonal entry of $A'A$ is equal to 9 if every column of A has norm 3.

Furthermore, if **a** and **b** have angle 60 degrees between them, then

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos 60°,$$

which is equal to 9/2 if **a** and **b** both have norm 3.

Therefore, $A'A$ has 9's along the diagonal and 9/2's everywhere else.

Problem 6

Find $\lim_{n\to\infty} \begin{bmatrix} 81 & 80 & -440 \\ -20 & -19 & 110 \\ 11 & 11 & -\frac{119}{2} \end{bmatrix}^n$, given the diagonalization

$$\begin{bmatrix} 81 & 80 & -440 \\ -20 & -19 & 110 \\ 11 & 11 & -\frac{119}{2} \end{bmatrix} = \begin{bmatrix} -15 & -16 & 80 \\ 4 & 5 & -20 \\ -2 & -2 & 11 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} -15 & -16 & 80 \\ 4 & 5 & -20 \\ -2 & -2 & 11 \end{bmatrix},$$

Solution

If $A = V\Lambda V^{-1}$, then $A^n = V\Lambda^n V^{-1}$. Taking $n \to \infty$, we get that

$$\Lambda^n \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore,

$$A^n \to \begin{bmatrix} -15 & -16 & 80 \\ 4 & 5 & -20 \\ -2 & -2 & 11 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -15 & -16 & 80 \\ 4 & 5 & -20 \\ -2 & -2 & 11 \end{bmatrix} = \begin{bmatrix} 161 & 160 & -880 \\ -40 & -39 & 220 \\ 22 & 22 & -120 \end{bmatrix}.$$