DATA 1010
In-class exercises
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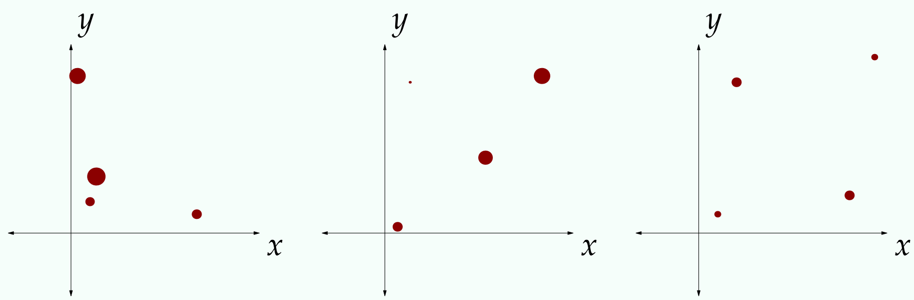
Problem 1

The first figure below shows the probability mass function for the joint distribution of two random variables X_1 and Y_1. The second and third figures show the joint distributions of (X_2, Y_2) and (X_3, Y_3).

For which value of i is $\mathbb{P}(Y_i > X_i)$ the largest?



Solution

The probability that Y is larger than X is the sum of the masses above the line $y = x$. Therefore, the probability is largest for for the first figure ($i = 1$).

Problem 2

Suppose that X is a random variable whose distribution has PMF $m_X(1) = 1/5$, $m_X(7) = 1/5$, and $m_X(\sqrt{3}) = 3/5$.

Suppose that Y is a random variable whose distribution has PMF $m_Y(1) = 1/4$, $m_Y(3) = 1/4$, $m_Y(11.5) = 1/4$, and $m_Y(-4) = 1/4$.

Suppose that X and Y are independent, and call their joint PMF $m_{(X,Y)}$. For how many ordered pairs (x, y) do we have $m_{(X,Y)}(x, y) > 0$?

Solution

For any pair (x, y), the probability that $X = x$ and $Y = y$ is the product of the probability that $X = x$ and the probability that $Y = y$. Therefore, $m_{(X,Y)}(x, y) > 0$ if and only if $m_X(x) > 0$ and $m_Y(y) > 0$. So there are $4 \times 3 = \boxed{12}$ points (x, y) where $m_{(X,Y)}(x, y) > 0$.

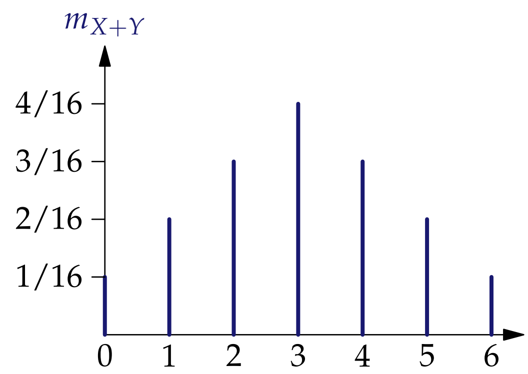
Problem 3

Suppose that X and Y are independent random variables whose distributions have constant probability mass functions on $\{0, 1, 2, 3\}$. Make a spike graph for the probability mass function of $X + Y$.

Solution

The probability that $X + Y = 0$ is equal to the product of the probabilities that X and Y are both zero. Therefore, it is $1/16$. The probability that $X + Y = 1$ is $2/16$, since it gets a $1/16$ probability mass from both of the events $\{X = 1, Y = 0\}$ and $\{X = 0, Y = 1\}$.

Continuing in this way, we find that the probability mass function increases in increments of $1/16$ up to $4/16$ and then decreases back down to $1/16$.



Problem 4

Show that if E and F are independent, then E and F^c are also independent.

Solution

We have

$$\mathbb{P}(E) = \mathbb{P}(E \cap F) + \mathbb{P}(E \cap F^c),$$

so

$$\mathbb{P}(E \cap F^c) = \mathbb{P}(E) - \mathbb{P}(E \cap F) = \mathbb{P}(E) - \mathbb{P}(E)\mathbb{P}(F),$$

by independence. Factoring out $\mathbb{P}(E)$, we get $\mathbb{P}(E \cap F^c) = \mathbb{P}(E)(1 - \mathbb{P}(F)) = \mathbb{P}(E)\mathbb{P}(F^c)$, as desired.

Problem 5

The 52 cards in a standard deck are shuffled and dealt out in four hands of 13 cards each. What is the conditional probability, given that the first two hands contain 8 of the 13 spades, that the fourth hand contains exactly 3 of the remaining spades?

Solution

We work directly with the reduced sample space. Once the first 26 cards are dealt and 5 spades remain, we can form a hand satisfying the given conditions by choosing 3 of the remaining 5 spades and 10 of the 21 non-spades to fill out the hand. Since these dealings are equally likely, the conditional probability is

$$\frac{\binom{5}{3}\binom{21}{10}}{\binom{26}{13}} \approx 0.339.$$