DATA 1010
In-class exercises
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Problem 1

Let $\Omega = \{(\texttt{H},\texttt{H}), (\texttt{H},\texttt{T}), (\texttt{T},\texttt{H}), (\texttt{T},\texttt{T})\}$, and let $m(\omega) = \frac{1}{4}$ for each $\omega \in \Omega$.

Identify a mathematical object in the model (Ω, m) which can be said to correspond to the phrase "the first flip turns up heads". Which of the following is true of this object?

(i) It is one of the values of the function m

(ii) It is the set Ω

(iii) It is a subset of Ω

(iv) It is one of the elements of Ω

Solution

The outcomes $(\texttt{H},\texttt{H})$ and $(\texttt{H},\texttt{T})$ are the ones which satisfy the condition "the first flip turns up heads". Therefore, the event corresponds to a **subset** of Ω, namely the subset $\{(\texttt{H},\texttt{H}), (\texttt{H},\texttt{T})\}$.

Problem 2

Explain how to obtain the probability of an event from the probability mass function.

For concreteness, consider $\Omega = \{(\texttt{H},\texttt{H}), (\texttt{H},\texttt{T}), (\texttt{T},\texttt{H}), (\texttt{T},\texttt{T})\}$, a probability mass function which assigns mass $\frac{1}{4}$ to each outcome, and the event $\{(\texttt{H},\texttt{H}), (\texttt{H},\texttt{T})\}$.

Solution

The probability of the event $\{(\texttt{H},\texttt{H}), (\texttt{H},\texttt{T})\}$ is the **sum** of the probabilities of the two outcomes in the event, namely $\frac{1}{4} + \frac{1}{4} = \frac{1}{2}$.

In general, we sum all of the probability masses of the outcomes in the event to find the probability of the event.

Problem 3

Match each term to its corresponding set-theoretic operation. Assume that E and F are events.

For concreteness, you can think about the events "first flip comes up heads" and "second flip comes up heads" for the two-flip probability space we've been considering.

(a) the event that E and F both occur	(i) the intersection $E \cap F$
(b) the event that E does not occur	(ii) the union $E \cup F$
(c) the event that either E occurs or F occurs	(iii) the complement E^{c}

Solution

The event that both E and F occur is $E \cap F$, since $E \cap F$ is the set of outcomes in both E and F.

The event that E does not occur is E^{c}, since the complement of E includes all the outcomes that are not in E.

The event that either E or F occurs is $E \cup F$, since $E \cup F$ is the set of outcomes which are in either E or F.

Problem 4

Suppose a group of n friends enter the lottery. For $i \in \{1, \ldots, n\}$ let E_i be the event that the ith friend wins. Express the following events using set notation.

1. At least one friend loses.
2. All friends win.
3. At least one friend wins.

Solution

1. The event that at least one friend loses is $\bigcup_{i=1}^{n} E_i^c$.
2. The event that all friends win is $\bigcap_{i=1}^{n} E_i$.
3. The event that at least one friend wins is $\bigcup_{i=1}^{n} E_i$.

Problem 5

What is the cardinality of the domain of the function $\mathbb{P}$ if

$$\Omega = \{(\mathtt{H}, \mathtt{H}), (\mathtt{H}, \mathtt{T}), (\mathtt{T}, \mathtt{H}), (\mathtt{T}, \mathtt{T})\}?$$

Solution

The domain of $\mathbb{P}$ is the set of subsets of Ω. Since Ω has 4 elements, there are $2 \times 2 \times 2 \times 2 = 16$ elements in the domain of $\mathbb{P}$.

Problem 6

Consider events A and B where the occurrence of A implies the occurrence of B. For example, suppose A is the event that the Red Sox outscore the Yankees by 5 runs or more, and let B be the event that the Red Sox win the game. Which of the following is the set-theoretic relationship between the events A and B?

(a) $A \cap B = \emptyset$

(b) $A \subset B$

(c) $B \subset A$

Solution

Every outcome in the event A is also in the event B. Therefore, we have $A \subset B$.

Problem 7

Use the additivity property and the fact that $A = (A \cap B) \cup (A \cap B^{\mathsf{c}})$ to show that if $B \subset A \subset \Omega$, then $\mathbb{P}(B) \leq \mathbb{P}(A)$.

Solution

We have $\mathbb{P}(A) = \mathbb{P}(A \cap B) + \mathbb{P}(A \cap B^c)$ by additivity. Since $A \cap B = B$ and probabilities are non-negative, it follows that

$$\mathbb{P}(A) = \mathbb{P}(B) + \mathbb{P}(A \cap B^c) \geq \mathbb{P}(B)$$

as required.

Problem 8

Show that $\mathbb{P}(A \cup B) \leq \mathbb{P}(A) + \mathbb{P}(B)$ for all events A and B.

Use this property to show that if A occurs with probability zero and B occurs with probability zero, then the probability that A *or* B occurs is also zero.

Solution

Define $\tilde{A}$ to be the set of ω's which are in A but not B, and let $\tilde{B}$ be the set of ω's which are in B but not A. Then

$$\mathbb{P}(A \cup B) = \mathbb{P}(\tilde{A} \cup \tilde{B} \cup (A \cap B)) = \mathbb{P}(\tilde{A}) + \mathbb{P}(\tilde{B}) + \mathbb{P}(A \cap B),$$

since $\tilde{A}$, $\tilde{B}$, and $A \cap B$ are disjoint and together make up $A \cup B$. Furthermore, since $\mathbb{P}(A) = \mathbb{P}(\tilde{A}) + \mathbb{P}(A \cap B)$ and similarly for B, we have

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) - \mathbb{P}(A \cap B) + \mathbb{P}(B) - \mathbb{P}(A \cap B) + \mathbb{P}(A \cap B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B) \leq \mathbb{P}(A) + \mathbb{P}(B),$$

as desired.

We have $\mathbb{P}(A \cup B) \leq \mathbb{P}(A) + \mathbb{P}(B) \leq 0 + 0 = 0$ if both A and B have probability zero, so $\mathbb{P}(A \cup B) = 0$ in that case.