**DATA 1010**
**In-class exercises**
**Samuel S. Watson**
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## Problem 1

Consider a collection of 165 human faces, in the form of 231 pixel × 195 pixel, grayscale images. If we regard each of these images as a vector in $\mathbb{R}^{45045}$ (by concatenating columns of each image), then the list of face images spans a 165-dimensional subspace of $\mathbb{R}^{45045}$.

Let $A$ be the $45045 \times 165$ matrix whose columns are the image vectors in the collection. Use the associated Jupyter notebook to view the first column of $U$ in the SVD $U\Sigma V'$ (rendered as a $231 \times 195$ grayscale image). Look at the second and third eigenvectors as well.

Show that each face in the collection can be recognizably reconstructed as a projection onto the span of the first several columns of $U$.

*Bonus: there is an extra face in the directory, if you want to test how well the eigenfaces can be used to reconstruct an image which is not part of the original set.*

## Problem 2

In the course text, the singular value decomposition of a non-square matrix $A$ has either $U$ or $V'$ non-square, depending on whether $m > n$ or $m < n$.

A different convention exists, wherein $U$ and $V'$ are square and $\Sigma$ has the same dimensions as $A$. In this convention, $\Sigma$ is diagonal in the sense that $\Sigma_{i,j} = 0$ whenever $i \neq j$.

Explain how to obtain the $\Sigma$-rectangular SVD in terms of the $\Sigma$-square SVD.

## Problem 3

Suppose $A$ is an $m \times n$ matrix. Use the singular value decomposition $A = U\Sigma V'$ (where $U$ and $V$ have orthonormal columns and $\Sigma$ is diagonal) to find the largest possible value of $|A\mathbf{x}|$ for any unit vector $\mathbf{x}$.

Hint: think about how each matrix $U$, $\Sigma$, and $V'$ affect the length of $\mathbf{x}$. You might want to use the $\Sigma$-rectangular form of the SVD.

## Problem 4

In the text we learned about differentiation rules for differentiable functions $f : \mathbb{R}^n \to \mathbb{R}^m$. However, there are also differentiation rules for matrix-valued functions of a real variable. Suppose that $A : \mathbb{R} \to \mathbb{R}^{m\times n}$ and $B : \mathbb{R} \to \mathbb{R}^{n\times p}$ (in other words, the entries of $A$ and $B$ depend on a single real parameter). Show that

$$\frac{\mathrm{d}}{\mathrm{d}t}(A(t)B(t)) = \frac{\mathrm{d}A(t)}{\mathrm{d}t}B(t) + A\frac{\mathrm{d}B(t)}{\mathrm{d}t}.$$

For example, we might have

$$A(t) = \begin{bmatrix} -2 & 4t \\ 8t^2 & t \end{bmatrix} \quad \text{and} \quad B(t) = \begin{bmatrix} 3-2t \\ -t \end{bmatrix}.$$