**DATA 1010**
**In-class exercises**
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## Whirlwind review of linear algebra

1. **Vectors**
   (a) A *linear combination* of a list of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is an expression of the form $c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n$.
   (b) The *span* of a list of vectors is the set of all vectors that can be written as a linear combination of them
   (c) A list of vectors is *linearly independent* if none of the vectors in the list is in the span of the others.
      i. Equivalently, a list is linearly independent if the only vanishing linear combination of the vectors is the trivial one (with all weights zero)
      ii. Equivalently, a list is linearly independent if no vector in the list is in the span of the vectors preceding it in the list
   (d) A *vector space* is a set of vectors which is closed under the vector operations (geometrically, the origin, or a line or plane or 3D space through the origin, etc.)
   (e) A *basis* of a vector space is a linearly independent spanning list.
   (f) Every linearly independent list is shorter than or equal in length to every spanning list.
   (g) Therefore, all bases of a given space have equal length. This length is called the *dimension* of the space.
   (h) Every linearly independent list can be extended to obtain a basis, and every spanning list can be trimmed to obtain a basis.
   (i) Given a vector space and a basis, every vector in the space can be written uniquely as a linear combination of the vectors in the basis. The weights in this linear combination are called the *coordinates* of the vector with respect to this basis.

2. **Linear transformations**
   (a) A *linear transformation* $T$ is a function from one vector space to another which distributes across scalar multiplication and addition: $T(\alpha\mathbf{v} + \mathbf{w}) = \alpha T(\mathbf{v}) + T(\mathbf{w})$ for all $\mathbf{v}$ and $\mathbf{w}$ in the domain of $T$.
   (b) Given a choice of basis for the domain and the codomain, linear transformations are in one-to-one correspondence with the set of $m \times n$ matrices, where $n$ is the dimension of the domain and $m$ the dimension of the codomain (the columns of this matrix contain the coordinates of the images of one basis with respect to the other).
   (c) The matrix product is defined to correspond to composition of linear transformations.
   (d) A linear transformation is fully specified by the images of the vectors in a basis.
   (e) The *rank* of a linear transformation is the dimension of its range.
   (f) The *null space* of a linear transformation is the set of vectors it maps to zero.
   (g) The *rank-nullity theorem* says that the rank of a linear transformation plus the dimension of its null space is equal to the dimension of the domain.

3. **Linear systems**.
   (a) We represent a linear system of equations in the form $A\mathbf{x} = \mathbf{b}$, where $A$ is an $m \times n$ matrix, $\mathbf{x} \in \mathbb{R}^n$, and $\mathbf{b} \in \mathbb{R}^m$.
   (b) $A\mathbf{x}$ can be interpreted as the linear combination of the columns of $A$ with weights given by the entries of $\mathbf{x}$
   (c) Therefore, $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $\mathbf{b}$ is in the span of the columns of $A$.
   (d) $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions if it has at least one and if the columns of $A$ are linearly independent.
   (e) If $A\mathbf{x} = \mathbf{b}$ has no solutions, then there is a unique vector in the range of $A$ whose distance to $\mathbf{b}$ is minimal.
   (f) If $A$ is $m \times n$ with $m < n$, then its columns must be linearly dependent. Therefore, $A\mathbf{x} = \mathbf{b}$ has either no solutions or infinitely many solutions.

(g) If $A$ is $m \times n$ with $n < m$, then its columns cannot span $\mathbb{R}^m$. Therefore, $A\mathbf{x} = \mathbf{b}$ has either no solutions or one solution.

4. **Matrix algebra and orthogonality**

    (a) If $A$ is a square matrix, then $\mathbf{x} \mapsto A\mathbf{x}$ is surjective if and only if it is injective.

    (b) If $A$ is square and $\mathbf{x} \mapsto A\mathbf{x}$ is bijective, its inverse $\mathbf{x} \mapsto A^{-1}(\mathbf{x})$ is linear

    (c) We have $(AB)^{-1} = B^{-1}A^{-1}$ if $A$ and $B$ are both invertible

    (d) The transpose $A'$ of a matrix $A$ is the matrix obtained by swapping rows and columns

    (e) We have $(AB)' = B'A'$ for any matrix product $AB$.

    (f) The *dot product* of two vectors measures their alignment: $\mathbf{x} \cdot \mathbf{y} = |\mathbf{x}||\mathbf{y}| \cos\theta$, where $\theta \in [0, 180°]$ is the angle between the two vectors.

    (g) Dot products can be calculated in terms of the transpose and matrix product, since $\mathbf{x} \cdot \mathbf{y} = \mathbf{x}'\mathbf{y}$.

    (h) The squared length of a vector is equal to the dot product of the vector with itself.

    (i) To find the component of a vector $\mathbf{v}$ in the direction of a unit vector $\mathbf{u}$, calculate $\mathbf{v} \cdot \mathbf{u}$.

    (j) A list of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is *orthogonal* if $\mathbf{v}_i \cdot \mathbf{v}_j = 0$ for any $i \neq j$.

    (k) For any list of vectors, there is an orthogonal list with the same span—work through the list and subtract from each vector its projection onto the span of the preceding ones (*Gram-Schmidt*)

    (l) A matrix $U$ is orthogonal if the columns of $U$ are orthogonal unit vectors.

        i. Equivalently, $U$ is orthogonal if $U'U = I$

        ii. If $U$ is square and orthogonal, then $UU' = I$. Otherwise, $UU' \neq U'U$ (in fact, $UU'$ is the matrix which projects a vector onto the span of the columns of $U$).

        iii. Geometrically, orthogonal matrices represent rotation/reflection—they are length- and angle-preserving

5. **Eigendecomposition**

    (a) An eigenvector $\mathbf{v}$ of an $n \times n$ matrix $A$ is a nonzero vector with the property that $A\mathbf{v} = \lambda\mathbf{v}$ for some $\lambda \in \mathbb{R}$ (in other words, $A$ maps $\mathbf{v}$ to a vector which is either zero or parallel to $\mathbf{v}$)

    (b) The eigenspace associated with a given eigenvalue $\lambda$ is the set of all solutions to the homogeneous sytsem $(A - \lambda I)\mathbf{v} = 0$, where $I$ is the $n \times n$ identity matrix

    (c) If a matrix has $n$ linearly independent eigenvectors (in other words, if the sum of the dimensions of the eigenspaces is equal to $n$), then $A$'s action on $\mathbb{R}^n$ can be understood simply. Each coordinate gets multiplied by the corresponding eigenvalue:

$$A(c_1\mathbf{v}_1 + \cdots c_n\mathbf{v}_n) = c_1\lambda_1\mathbf{v}_1 + c_n\lambda_n\mathbf{v}_n.$$

**Problem 1**

Show that for any $m \times n$ matrix $A$, the matrices $A'A$ and $A$ have the same null space and therefore also the same rank.

**Problem 2**

In Julia, you can divide each entry of a matrix $M$ by a number $x$ using the expression `M ./ x`.

Write a function `mydiv` from scratch to perform the same operation.

## Problem 3

Visit `https://shadanan.github.io/MatVis/` and set the matrix to

$$\begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$$

Look for the eigenvectors of this matrix *visually* as you manipulate the slider. That is, look for vectors (with tails at the origin) that are not being turned at all by the matrix. Turn on the "show eigenvectors" checkbox to check your answer (or use Julia).

Repeat this exercise with

$$\begin{bmatrix} -2 & 1 \\ -1 & 2 \end{bmatrix} \text{ and } \begin{bmatrix} 2 & 1 \\ -2 & 2 \end{bmatrix}$$